HYDROMAGNETIC WAVES IN A BUOYANT MEDIUM

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by

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ABSTRACT

Lighthill has suggested that magnetic fields can transform gravity waves into Alfvén waves. The behaviour of gravity waves in an incompressible and infinitely conducting atmosphere is examined so as to check this suggestion. It is found that an upgoing gravity wave is converted to a downgoing Alfvén wave and that this conversion is affected without any introduced atmospheric discontinuity. In the ionosphere dissipation is sufficient, save at very long wavelengths, to inhibit mode conversion but in the solar atmosphere the process may well operate. The energy necessary to heat the solar corona could possibly be supplied by gravity waves when dissipation occurs after the conversion to an Alfvén wave.

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CHAPTER 1

GRAVITY WAVES IN NEUTRAL AND IONISED ATMOSPHERES

Gravity waves are waves which propagate within a fluid medium. They propagate by means of an interchange of energy between the kinetic energy of motion and the potential energy gained by deforming a gravitationally stratified fluid. As an internal wave, the velocity field and hence the wave's kinetic energy must be defined throughout the fluid; the potential energy ultimately depends upon the scale of the fluid's stratification and must be similarly defined. This scale is sufficient impediment to laboratory study. It is therefore not surprising that interest in these waves awaited observations of fluid motions on an apt scale. The importance of internal gravity waves in the atmosphere was first suggested by ionospheric observations. Here the velocity field throughout an extended region of the atmosphere is inferred from the perturbations it induces in the electron density. As a consequence atmospheric gravity wave theory has until recently followed the direction dictated by the ionospheric observations it seeks to explain.

Electron densities in the ionosphere show considerable variability on a time scale of about one hour. Indeed perturbations on this time scale are a major ionospheric feature, being exceeded in amplitude by only the fundamental diurnal variation and an occasional intense ionospheric storm. It is difficult to find ionospheric observations which do not show some indication of such perturbations. If these perturbations represented only a small fractional change in the electron density, then there seems little point to study them beyond the identification of their physical cause. They would be little more than of nuisance value on an otherwise smooth ionosphere. Both their amplitude and frequency of occurrence belie this. There is now a sufficient body of evidence to identify at least some parts, if not all, of the perturbation spectrum with internal gravity waves.

In a stable atmosphere, gravity waves transport energy. They may also transport both linear and angular momentum. They appear as a principal mechanism by which an atmosphere seeks to regain equilibrium from a disturbed state. Their role in a stable atmosphere parallels that of convection in an unstable atmosphere and hence they are of fundamental scientific interest.

The differential equation governing the behaviour of gravity waves propagating in a perfect fluid without dissipation is derived under the assumption that perturbations are small from the fluid's equations of motion and the equation of state. This artificiality makes the mathematics tractable and leads to solution of wave propagation under a wide range of physical conditions. Variations imposed on the ambient atmosphere appear as variable coefficients in a second order equation of the classical wave equation type. The effects of ambient temperature variations and wind shear on wave propagation are well documented in literature. These lead to delightful wave properties such as reflection, ducting and critical wave behaviour. The latter process mirrors the inadequacy of both the fluid model and the perturbation method but nevertheless gives insight into behaviour under more realistic conditions.

The effects of viscosity may also be treated as perturbations upon this second order wave equation and yield valuable information about the behaviour when dissipation is small. However, Yanowitch (1967) has derived an equation in which the viscous effects are unbounded. This equation is a fourth order wave equation and as such yields properties somewhat different to those deduced from small viscosity. One paramount difference exists. Yanowitch's solution gives wave perturbations which are bounded. The wave is reflected by viscous interaction even though the reflection coefficient is small, save at long vertical wavelengths. Solutions may thus always be made compatible with the postulates of the perturbation method. In physical terms this implies that the upper boundary of the atmosphere is non-radiative. The pole at infinity which exists in some solutions for a perfect fluid is removed by the inclusion of realistic dissipation.

In a conducting atmosphere with an imposed magnetic field, Lorentz forces operate in addition to the gravitational forces. The behaviour of gravity waves is therefore modified. Lighthill (1967) has considered this and with particular reference to the solar chromosphere states; 'Magnetic fields can transform gravity waves into Alfvén waves at high altitudes preventing their reflection from regions of increasing temperature'. Novel and surprising behaviour is therefore possible.

The purpose of this thesis is threefold. Firstly to investigate this effect originally suggested by Lighthill. We take the simplest possible atmospheric model which retains only those features essential to its preservation. To this end we assume;

(a) an incompressible fluid,

(b) infinite conductivity,

(c) an isothermal atmosphere in hydrostatic equilibrium

(d) a uniform ambient magnetic field.

Howe (1969) has given the derivation of the relevant differential equations governing the motions in terms of the fluid's vorticity and its displacement; both these quantities being measured in the vertical direction. Here we rederive these equations, and treat them as independent wave equations thereby leading to a different physical interpretation.

The second purpose is to find the general behaviour of gravity waves in ionized media. How dissipation affects their propagation, and to within what limits the more exotic behaviour applies. And finally to learn from the advances made in the study of hybrid waves in the solar atmosphere to see if these may be applied to the ionosphere.

P A R T I

WAVES IN A PERFECT HYDROMAGNETIC MEDIUM

CHAPTER 2

THE EQUATIONS OF MOTION

Following Chandrasekhar (1961, p.147) the equations of motion for a conducting fluid are,

$$\rho \frac{D\underline{u}}{Dt} = - \nabla p + \underline{\mathcal{L}} + \underline{f}, \tag{2.1}$$

where $\underline{\mathcal{L}}$ is the Lorentz force per unit volume expressing the interaction between the conducting fluid and the imposed magnetic field; and $\underline{f}$ is the imposed body force of mechanical origin. The body forces will here be restricted to gravitational forces only. To this equation we add the equation expressing the conservation of matter,

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0, \tag{2.2}$$

and further, if the fluid is incompressible

$$\nabla\cdot\underline{u} = 0. \tag{2.3}$$

For a conducting fluid where the displacement current is neglected, Maxwell's equations assume the form,

$$\nabla\times\underline{H} = \underline{J} \tag{2.4}$$

$$\nabla\times\underline{E} = -\frac{\partial\underline{B}}{\partial t} \tag{2.5}$$

$$\nabla\times\underline{H} = 0. \tag{2.6}$$

To these equations we add Ohm's Law

$$J = \sigma(\underline{E} + \underline{u}\times\underline{B}) \tag{2.7}$$

The Lorentz force is given by,

$$\underline{\mathcal{L}} = \underline{J} \times \underline{B}, \tag{2.8}$$

where

$$\underline{\mathcal{L}} = \mu(\nabla \times \underline{H}) \times \underline{H} \tag{2.9}$$

Now eliminating $\underline{E}$ between equations (2.5) and (2.7)

$$\frac{\partial \underline{B}}{\partial t} + \nabla \times \{\frac{1}{\sigma} \underline{J} - \underline{u} \times \underline{B}\} = 0. \tag{2.10}$$

We now choose to work in the infinitely conducting limit. This assumption allows considerable simplification and as will be shown later gives some separation of modes that does not occur in a more general treatment. Equation (2.10) reduces to

$$\frac{\partial \underline{H}}{\partial t} = \nabla \times (\underline{u} \times \underline{H}), \tag{2.11}$$

or with some further simplification together with equation (2.7),

$$\frac{D\underline{H}}{Dt} = (\underline{H} \cdot \nabla)\underline{u}. \tag{2.12}$$

The dynamical equations together with equation (2.9) and equation (2.12) will be seen to form a consistent set of eight equations in eight unknowns ($\underline{u}$, $\underline{H}$, p, ρ). These in principle may be solved.

CHAPTER 3

THE PERTURBATION EQUATIONS AND A WAVE EQUATION

Consider an application where the mean fluid density is horizontally stratified and exponentially decreasing with height, the mean flow is zero and the imposed magnetic field is uniform and inclined at an arbitrary angle θ to the horizontal. The equations of the previous section yield their following perturbation equivalents.

$$\rho_0 \frac{\partial \underline{u}}{\partial t} - \mu(\nabla \times \underline{h}) \times \underline{H} = - \nabla p - \rho \underline{g} \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + (\underline{u}\cdot\hat{k}) \frac{d\rho_0}{dz} = 0 \tag{3.2}$$

$$\nabla\cdot\underline{u} = 0 \tag{3.3}$$

$$\frac{\partial \underline{h}}{\partial t} = (\underline{H}\cdot\nabla)\underline{u}, \tag{3.4}$$

where $\hat{k}$ is a unit vector in the vertical direction,

$\underline{H}$ is the imposed uniform field

and $\underline{h}$ is the perturbation magnetic field.

The terms u, p, ρ now denote perturbation variables while ρ_0 is the unperturbed fluid density. Eliminating $\underline{h}$ and ρ from equation (3.1), then

$$\rho_0\left(\frac{\partial \underline{u}}{\partial t^2} + N^2(\underline{u}\cdot\hat{k})\hat{k}\right) - \rho_0 c_a{}^2 \hat{\underset{\cdot}{s}}\cdot\nabla(\nabla \times \underline{u}) \times \hat{s} = -\nabla \frac{\partial \underline{p}}{\partial t}, \tag{3.5}$$

where $\underline{H} = H\,\hat{s}$

$N^2 = -\dfrac{g}{\rho_0}\dfrac{d\rho_0}{dz}$, the Brunt frequency squared, and $c_a^{\,2} = \frac{\mu H^2}{\rho_0}$, the Alfvén velocity SQUARED.

Without loss in generality one may write

$$\hat{S} = (\cos\theta,\ 0,\ \sin\theta) \tag{3.6}$$

$$\underline{u} = (u,\ v,\ w) \tag{3.7}$$

$$\hat{S}\cdot\nabla = \frac{\partial}{\partial S}\ . \tag{3.8}$$

Equation (3.8) represents the spatial derivative of a dependent variable along the direction of the imposed magnetic field. Equation (3.5) written in its expanded form becomes,

$$\frac{\partial^2 u}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial t} + c_a^{\,2} \sin\theta \frac{\partial}{\partial S}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{3.9}$$

$$\begin{aligned}\frac{\partial^2 v}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial y \partial t} &+ c_a^{\,2} \cos\theta \frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\ &- c_a^{\,2} \sin\theta \frac{\partial}{\partial S}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\end{aligned} \tag{3.10}$$

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} - c_a^{\,2} \cos\theta \frac{\partial}{\partial S}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{3.11}$$

These equations together with $\nabla\cdot\underline{u} = 0$ describe the possible motions of the medium. Elimination of any three dependent variables gives a differential equation which is sixth order in time. By assuming solutions of a sinusoidal form, this equation may be interpreted in terms of either propagating or evanescent waves. However, the primative equations contain time derivatives in only the second order and we seek three modes each of which, if propagating, will have two possible directions of travel. A separation into three independent wave equations, should it occur, indicates three independent and non-interacting wave modes. Alternatively, if a wave equation is derived whose terms contain the effects of both gravitational and magnetic forces, that is one whose coefficients contain both the Brunt frequency and the Alfvén velocity, then waves of mixed character are possible.

Now, eliminating p between equations (3.9) and (3.10) yields immediately,

$$\frac{\partial^2 \omega}{\partial t^2} - c_a^2 \frac{\partial^2 \omega}{\partial s^2} = 0, \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.12}$$

For the moment consider $\omega \neq 0$. Equation (3.12) is an independent wave equation which contains only the Alfvén velocity and represents a wave progressing in either direction along the magnetic field line with this velocity. Equation (3.12) is alternatively derived from the primative equations by putting $w = 0$, motion is thus at all times directed horizontally and gravitational forces do not operate. For plane waves, the horizontal motion is perpendicular to the horizontal projection of the wave vector. This wave may therefore be identified with the classical transverse Alfvén mode which propagates in an unstratified incompressible fluid. On the other hand, putting $\omega = 0$ satisfies equation (3.12). Therefore putting $\omega = 0$ into equations (3.9) to (3.11) suppresses the slow Alfvén mode and enables us to derive an equation for the remaining two modes.

Since the vertical component of vorticity is zero for the remaining two modes, we may use this to eliminate u and v in turn from the equation describing the fluid's incompressibility, $\nabla \cdot \underline{u} = 0$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial z} = 0 \tag{3.13}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z} = 0 \tag{3.14}$$

This then allows the elimination of u from equation (3.9) to give,

$$\frac{\partial^3 w}{\partial z \partial t^2} = \frac{1}{\rho_0} \frac{\partial}{\partial t} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) + c_a^{\,2} \sin \theta \frac{\partial}{\partial s} \nabla^2 w \tag{3.15}$$

while finally p may be eliminated by the use of equation (3.11)

$$\frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\rho_0 \frac{\partial^3 w}{\partial z \partial t^2} \right) - c_a^{\,2} \frac{\partial^2}{\partial s^2} \nabla^2 w + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2 w}{\partial t^2} + N^2 w \right) = 0 \tag{3.16}$$

Clearly, if $c_a^{\,2} = 0$, then this equation is that of a gravity wave in an incompressible and stratified fluid, while the asymptotic approximation gained by putting $N^2 = 0$ yields the equation

$$\left(\frac{\partial^2}{\partial t^2} - c_a^{\,2} \frac{\partial^2}{\partial s^2} \right) \nabla^2 w = 0 \tag{3.17}$$

This limiting form of Alfvén wave allows motion along the magnetic field line and represents a differing polarization from that given by equation (3.12).

It will be noted that equation (3.16) does not factor into two separate second order wave equations. The resulting fourth order differential equation thus represents modes that are a mixture of both gravity and magnetic waves.

We require an expression for the energy transported by the wave motion. Following the standard method of analysis we form, from equation (3.1) and equation (3.4), the sum

$$\rho_0 \underline{u} \cdot \frac{\partial \underline{u}}{\partial t} + \mu \underline{n} \cdot \frac{\partial \underline{n}}{\partial t} = \mu \left[(\nabla \times \underline{h}) \times \underline{H} \right] \cdot \underline{u} - \underline{u} \cdot \nabla p - \rho \underline{g} \cdot \underline{u} + \mu \underline{h} \cdot (H \cdot \nabla) \underline{u} \qquad (3.18)$$

Using equation (3.4) this may be simplified to give,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_0 \underline{u} \cdot \underline{u} + \tfrac{1}{2}\mu \underline{h} \cdot \underline{h} + \tfrac{1}{2}\rho_0 N^2 \zeta^2) + \nabla \cdot (p\underline{u}) + \nabla \cdot \left[\rho_0 c_a^{\ 2} \frac{\partial(\underline{\xi} \cdot \hat{S})}{\partial S} \underline{u} \right] - \frac{\partial}{\partial S} \left[\rho_0 c_a^{\ 2} \underline{u} \cdot \frac{\partial \underline{\xi}}{\partial S} \right] = 0 \qquad (3.19)$$

where $\underline{\xi}$ is the fluid displacement vector

and ζ is the vertical component of this vector.

Since we are interested in wave motions, quantities of prime concern are the mean energy densities and the mean energy fluxes where this mean may be conveniently taken over the time of one wave cycle. All terms in equation (3.19) are of the second order in the perturbation variables and in general no term of this equation identically vanishes when averaged in this way. Averaging will be denoted by an overbar in the subsequent discussion. Now equation (3.19) has the form of a continuity equation relating the partial time derivative of the total energy density to the spatial derivatives of terms which are dimensionally an energy flux. Clearly the term $p\underline{u}$ is a mechanical flux term. The 'magnetic' flux is represented by two terms which now require interpretation.

Integrating equation (3.4) with respect to time gives

$$\underline{h} = H \frac{\partial \underline{\xi}}{\partial s} \tag{3.20}$$

The integration constant is omitted as its product with any other perturbation variable will generate a first order term having a zero mean value. Using this expression, then equation (3.19) may be rewritten as,

$$\frac{\partial}{\partial t} (\tfrac{1}{2}\rho_0\overline{\underline{u}\cdot\underline{u}} + \tfrac{1}{2}\mu\overline{\underline{h}\cdot\underline{h}} + \tfrac{1}{2}\rho_0\overline{N^2\zeta^2}) + \nabla\cdot(\overline{p+\mu\underline{H}\cdot\underline{h})\underline{u}}$$

$$- \nabla\cdot\overline{(\underline{u}\cdot\underline{h})\mu\underline{H}} = 0 \tag{3.21}$$

The second term of the magnetic flux is expressed as a divergence by virtue of the relationship,

$$H\cdot\nabla\phi = \nabla\cdot\phi\underline{H}; \quad \nabla\cdot H = 0. \tag{3.22}$$

The magnetic flux is now divided into two quantities. The first term contains a linearized factor expressing the increase of magnetic pressure due to the magnetic perturbation. This adds directly to the fluid pressure perturbation and increases the effective mechanical flux. On the other hand, the second term depends on the magnetic tension $\mu H\underline{H}$ and this term is associated with the stretching of the magnetic lines of force. It generates a negative flux. Using the identity,

$$\underline{a}\times(\underline{b}\times\underline{c}) \equiv (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c} \tag{3.23}$$

then,

$$\frac{\partial}{\partial t} (\tfrac{1}{2}\rho_0\overline{\underline{u}\cdot\underline{u}} + \tfrac{1}{2}\mu\overline{\underline{h}\cdot\underline{h}} + \tfrac{1}{2}\rho_0\overline{N^2\zeta^2}) + \nabla\cdot[\overline{p\underline{u}} + \mu\overline{\underline{h}\times(\underline{u}\times\underline{H})}] = 0 \tag{3.24}$$

The flux term $\mu \underline{h} \times (\underline{u} \times \underline{H})$ can be identified with the Poynting vector.

If the fluid has a velocity $\underline{u}$, then the electric field it will experience is not that measured by a stationary observer, but

$$\underline{E}' = \underline{E} + \mu\, \underline{u} \times \underline{H}. \tag{3.25}$$

The current density is given by the relation

$$\underline{J} = \sigma E', \tag{3.26}$$

and in the limit of infinite conductivity $\underline{E}' = 0$,

$$E = -\mu\, \underline{u} \times \underline{H}. \tag{3.27}$$

The second term contained within the divergence term of equation (3.24) is then

$$\mu\, \overline{h \times (\underline{u} \times \underline{H})} = \overline{\underline{E} \times \underline{h}}$$

and is the Poynting vector averaged over one wave cycle.

CHAPTER 4

HARMONIC WAVES

Now confining our attention to harmonic waves. When the medium is stratified in the vertical direction only, the partial differential equation describing the motion can be separated into equations each one of which contains derivatives in only one of the independent variables. The variables of separation in the equations associated with the independent variables x, y and t are identified with the horizontal components of the wavenumber (k_x and k_y) and the frequency of the wave σ. The functional form of w correspon-ding to this separation is

$$w = w(z)\ \exp i(k_x \cdot x + k_y \cdot y - \sigma t) \tag{4.1}$$

and waves satisfying this form of separation are called harmonic waves. In this case equation (3.16) reduces to the form

$$\left(k_x^2 + \frac{\sigma_a^2}{\sigma^2}\frac{\partial^2}{\partial S^2}\right)\left(\frac{\partial^2 w}{\partial z^2} - k_h^2 w\right) + \frac{k_x^2}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial w}{\partial z} + k_x^2 k_h^2 \frac{N^2}{\sigma^2} w = 0 \tag{4.2}$$

where the Alfvén frequency is defined by $\sigma_a^2 = c_a^2 k_x^2$. The operator $\frac{\partial}{\partial S}$ expressed in terms of other derivatives is

$$\frac{\partial}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial}{\partial x} + \frac{\partial z}{\partial S}\frac{\partial}{\partial z} \tag{4.3}$$

which in terms of the wave numbers reduces to

$$\frac{\partial}{\partial S} = ik_x \cos\theta + \sin\theta \frac{\partial}{\partial z} \tag{4.4}$$

Equation (4.4) can now be written in terms of total derivatives in z, and is

$$\left[k_x^2 + \frac{\sigma_a^2}{\sigma^2}\left(ik_x \cos\theta + \sin\theta \frac{d}{dz}\right)^2\right]\left[\frac{d^2w}{dz^2} - k_h^2 w\right]$$

$$+ k_x^2 \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{dw}{dz} + k_x^2 k_h^2 \frac{N^2}{\sigma^2} w = 0 \tag{4.5}$$

In a stratified medium where the density decreases exponentially as $z/\mathcal{H}$, the coefficient $\frac{1}{\rho_0}\frac{d\rho_0}{dz}$ can be removed from the differential equation. The transformations $z = z'\mathcal{H}$, $k_x = k_x'/\mathcal{H}$ etc. changes the logarithmic density gradient to -1 and adds primes to other coefficients containing a dimension of length. The scale of the density decrease $\mathcal{H}$ is analogous to the atmospheric scale height and for convenience will be called the 'scale height'. Except where explicitly stated all parameters will be regarded as being normalized by $\mathcal{H}$ and the prime signifying this normalization will be dropped.

Further approximations may be considered which neglect part or all of the curvature of the contours of equal phase in an assumed wave-like solution. These approximations can be considered as 'local' plane harmonic wave approximations. Let w vary sinusoidally with a vertical wave number k_z in the manner,

$$w = W(z)\, e^{ik_z(z)\cdot z} \tag{4.8}$$

If derivatives of $W(z)$ and k_z are locally small then

$$\frac{d_n w}{dz^n} = (ik_z)^n w \tag{4.9}$$

Terms of the order of $\frac{1}{W}\frac{dW}{dz}$ and $\frac{1}{k_z}\frac{dk_z}{dz}$ are here smaller than unity. Application of this approximation leads to an algebraic equivalent of the differential equation.

One further approximation needs to be considered, that is the Boussinesq approximation. Before normalizing the differential equation contained the term $\frac{1}{\rho_0}\frac{d}{dz}(\rho_0 \frac{du}{dz})$. Write

$$\frac{1}{\rho_0}\frac{d}{dz}(\rho_0 \frac{dw}{dz}) \neq \frac{d}{dz}\left|\frac{dw}{dz} + \left(\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right) w\right| \approx \frac{d_2 w}{dz^2}$$

$$\text{if } \frac{1}{\rho_0}\frac{d\rho_0}{dz} << \frac{kdw}{wdz} \tag{4.10}$$

The equivalent plane harmonic wave approximation is

$$-k_z^{\ 2} + \frac{1}{\rho_0}\frac{d\rho_0}{dz} \cdot ik_z \approx -k_z^{\ 2}; \quad \frac{1}{\rho_0}\frac{d\rho_0}{dz} << k_z \tag{4.11}$$

This approximation supposes that the vertical wavelength is short compared to the scale height of the medium and is called the Boussinesq approximation.

By approximating $dw/dz = ik_z$ equation (4.2) defines a dispersion relationship. Only in the Boussinesq limit are all coefficients of this relation real and discussion in this section is restricted to this limit.

$$k^2(\sigma^2 - \sigma_a^{\ 2}\frac{k_s^{\ 2}}{k_x^{\ 2}}) = N^2 k_h^{\ 2}, \tag{4.12}$$

where

$$k_s = k_x \cos\theta + k_z \sin\theta. \tag{4.13}$$

For the particular case where $\sigma_a^2 = 0$ then this dispersion equation is identical to that of a gravity wave in an incompressible and non-conducting medium while if the imposed magnetic field is directed horizontally it takes the simple quadratic form

$$(\sigma^2-\sigma_a^2)k_z^2 = (N^2-\sigma^2+\sigma_a^2)k_h^2. \tag{4.14}$$

However, when the field is directed vertically, $k_s = k_z$ and the dispersion relationship is quadratic in k_z^2 yielding roots,

$$k_z^2 = k_x^2 \frac{(\sigma^2-\sigma_a^2) \pm [(\sigma^2-\sigma_a^2)^2 - 4(N^2-\sigma^2)\sigma_a^2]^{\frac{1}{2}}}{2\sigma_a^2} \tag{4.15}$$

When $\sigma^2 >> \sigma_a^2$ this expression has asymptotes

$$k_z^2 = k_x^2 \frac{\sigma^2}{\sigma_a^2}; \tag{4.16}$$

a magnetic asymptote representing both up-going and down-going magnetic waves, and

$$k_z^2 = k_x^2 (\frac{N^2}{\sigma^2} - 1); \tag{4.17}$$

a gravity wave limit representing waves propagating in either direction.

The behaviour of the roots when θ is small can be found by writing $\cos\theta \approx 1$, $\pm \sin\theta = \varepsilon$. Thus

$$\frac{\sigma_a^2}{\sigma^2}\left(1 + \varepsilon\frac{k_z}{k_x}\right)^2 = 1 - \frac{N^2}{\sigma^2}\frac{k_h^2}{k^2} \tag{4.18}$$

giving

$$\varepsilon = \frac{k_x}{k_z}\left[\pm\,\frac{\sigma}{\sigma a}\left(1 - \frac{N^2}{\sigma^2}\,\frac{k_h^{\,2}}{k^2}\right)^{\frac{1}{2}} - 1\right] \tag{4.19}$$

The expression contained within the square brackets is identically zero when the dispersion equation for $\theta = 0$ is inserted. On the other hand making $\pm\, k_z$ large ensures ε tends to zero. Two of the roots of the biquadratic equation (4.12) therefore diverge for all values of σ_a/σ as θ approaches zero.

It appears that the ratio $\sigma_a^{\,2}/\sigma^2$ is a fundamental parameter of the wave motion: that is the ratio of the Alfvén frequency to the wave frequency. Equation (4.2) and its associated dispersion relationship may be further parameterized. In an isothermal atmosphere where the atmospheric density decreases exponentially with the reduced height 'z', we may take the level $z = 0$ as that level where $\sigma_a^{\,2}/\sigma^2 = 1$. Plots of k_x versus z are found in Figure (4.1). The solid curve refers to $\theta = 90^{\circ}$ while a typical example of the variation of k_z versus the reduced height when the magnetic field is inclined at an angle to the horizontal is shown by the dashed curves.

The curves for a zenithal magnetic field are closed, the gravity mode is smoothly connected to that branch of the curve describing the Alfvén mode. This is not obviously true when the magnetic field is inclined. The complete dispersion equation is,

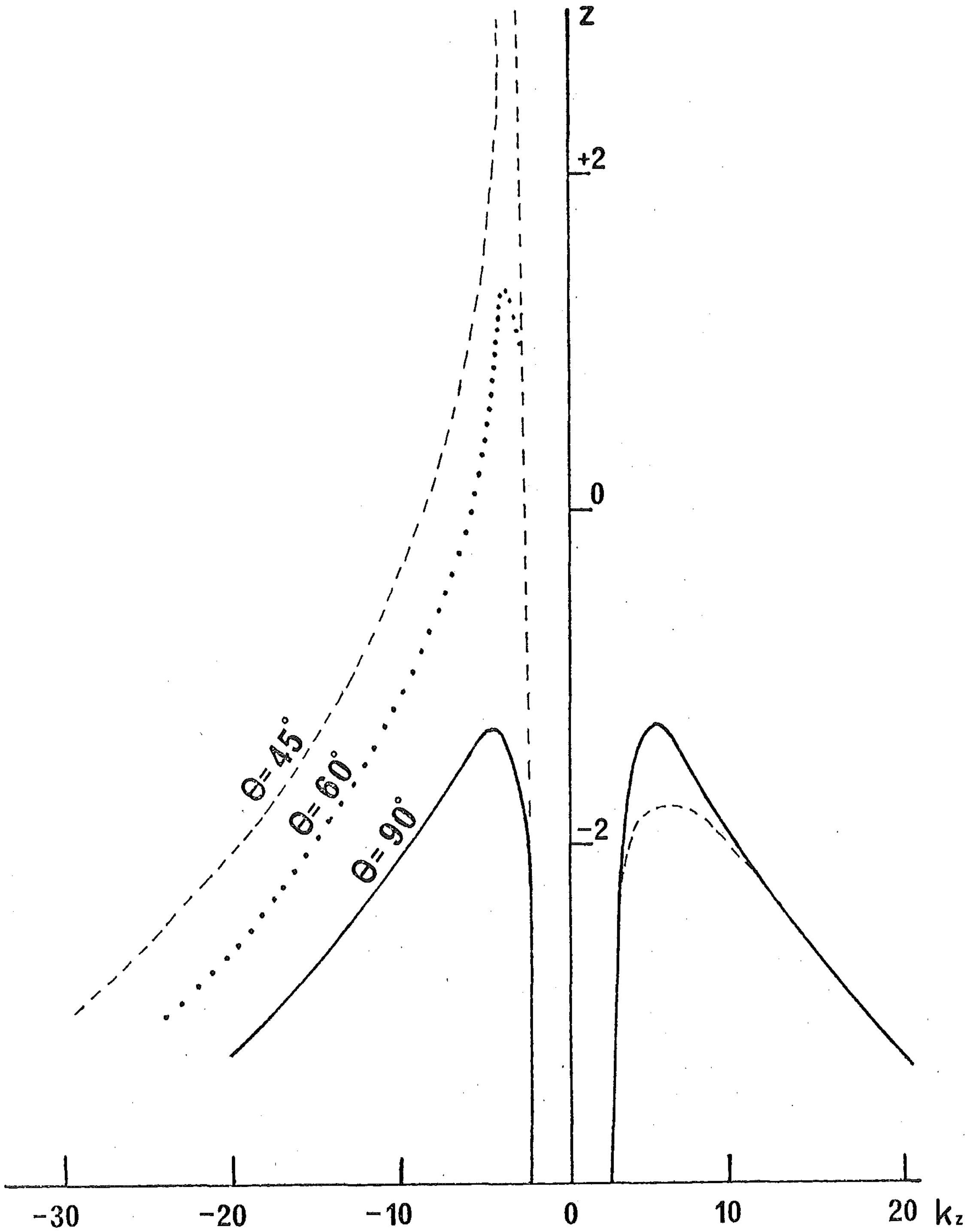

FIGURE 4.1 The dispersion curves under the Boussinesq approximation, plotting the variation of the vertical wave number k_z in an exponential atmosphere with applied uniform magnetic field. The height scale z denotes the reduced height satisfying the relationship $z = \log_e(c_a^{2}k_x^{2}/\sigma^2)$. Wave parameters are $k_x = 4.25$, $k_y = 0.0$, $N/\sigma = 1.2$. The solid curve gives the variation when the applied magnetic field is directed in the vertical direction while the dashed curve shows the variation when the field is inclined at an angle of 45°.

$$\left(\frac{k_z^2}{k_x^2} + \frac{k_n^2}{k_x^2}\right)\left[\left(\frac{k_z}{k_x}\sin\theta + \cos\theta\right)^2 - \frac{\sigma^2}{\sigma_a^2}\right] + \frac{k_h^2}{k_x^2}\frac{N^2}{\sigma^2}\frac{\sigma^2}{\sigma_a^2} = 0 \tag{4.20}$$

Now at great heights σ^2/σ_a^2 is small and in the limit where this tends to zero the four roots of equation (4.20) tend to the limit $k_z = \pm ik_h$, $-k_x\cot\theta$, $-k_x\cot\theta$. Expanding these roots to a higher approximation in σ/σ_a gives

$$k_z = \pm ik_h\left[1 + \frac{\frac{N^2}{\sigma^2}\frac{2}{\sigma_a^2}}{(\cos\theta \pm i\frac{k_x}{k_h}\sin\theta)^2}\right] \tag{4.21}$$

$$k_z = -k_x\left\{\cot\theta \pm \frac{\sigma}{\sigma_a}\left[\frac{\cot^2\theta - \frac{k_h^2}{k_x^2}\left(\frac{N^2}{\sigma^2} - 1\right)}{\cot^2\theta + \frac{k_h^2}{k_x^2}}\right]^{\frac{1}{2}}\right\} \tag{4.22}$$

A proof of these approximations is given in Appendix II. We note that the continuation of the roots corresponding to $k_z = \pm ik_h$ are always complex and thus represent evanescent modes. On the other hand those corresponding to $k_z = -k_x\cot\theta$ are complex only for

$$\frac{k_h^2}{k_x^2}\left(\frac{N^2}{\sigma^2} - 1\right) > \cot^2\theta$$

and real for

$$\frac{k_h^2}{k_x^2}\left(\frac{N^2}{\sigma^2} - 1\right) < \cot^2\theta.$$

The dispersion equation in the Boussinesq approximation thus indicates that waves may penetrate to $z = \infty$ when $\frac{k_z'}{k_x} < \cot\theta$,

k_z' being the vertical wave number of the asymptotic gravity wave. That is, all waves whose initial trajectories lie above a plane containing the magnetic field lines will in this approximation penetrate to infinity.

Equation (4.22) implies that $k_s = 0$ for small values of σ/σ_a. Now putting $k_s \simeq 0$ in the dispersion equation reduces it to a form which describes a gravity wave unaffected by the presence of the magnetic field. In the limiting case where $k_s = 0$ for all σ/σ_a the equation is identical with that of a gravity wave. Lighthill (1967) suggested that the penetrating wave was magnetic in character. Some doubt therefore exists and this can only be resolved by closer examination of wave behaviour in the Boussinesq approximation or by taking higher order approximations.

Provided k_z is real, the group velocity may be derived from the dispersion equation by the use of the expression

$$\underline{v}_g = \frac{\partial\sigma}{\partial\underline{k}} \tag{4.23}$$

Thus,

$$v_{gx} = \frac{c_a^2 k_s}{\sigma}\cos\theta + \frac{N^2-\sigma^2+c_a^2k_s^2}{k^2}\cdot\frac{k_x}{\sigma} \tag{4.24}$$

$$v_{gy} = \frac{N^2-\sigma^2+c_a^2k_s^2}{k^2}\cdot\frac{k_y}{\sigma} \tag{4.25}$$

$$v_{gz} = \frac{c_a^2 k_s}{\sigma}\sin\theta - \frac{\sigma^2-c_a^2k_s^2}{k^2}\cdot\frac{k_z}{\sigma} \tag{4.26}$$

whence we derive

$$\underline{v}_g \cdot \frac{k}{\sigma} = \frac{c_a^2 k_s^2}{\sigma^2} \tag{4.27}$$

or expressed in terms of the phase velocity

$$\underline{v}_g \cdot \underline{v}_p = c_a^2 \frac{k_s^2}{k^2} \tag{4.28}$$

Deep in the atmosphere the group velocity of the gravity like mode is asymptotic to that of a gravity wave in a non-conducting medium; the group and phase velocities are orthogonal and equation (4.27) is satisfied. On the other hand the magnetic mode is typified by large values of k_z. The dispersion equation requires that in this limit

$$1 - \frac{c_a^2 k_s^2}{\sigma^2} = 0\left(\frac{1}{k^2}\right) \tag{4.29}$$

The first terms in the expressions for $v_{gx,z}$ thus dominate. The asymptotic behaviour of the magnetic mode is consequently,

$$\underline{v}_g = c_a \hat{s} \tag{4.30}$$

being the Alfvén velocity directed along the magnetic field line.

From the dispersion relationship we may deduce that,

$$\frac{\partial c_a^2}{\partial k_z} = - \frac{\partial \sigma}{k_s^2} v_{gz}. \tag{4.31}$$

Now identifying c_a^2 as a height parameter, then the maximum on the k_z versus z plot shown in Figure (4.1) corresponds to a zero vertical group velocity. In this case the dispersion equation gives a well behaved group velocity. For σ/σ_a small the dispersion equation is the approximation given by equation (4.22) and is

$$\left(\frac{k_z}{k_x} + \cot\theta\right)^2\left(\cot^2\theta + \frac{k_x^{\,2}}{k_h^{\,2}}\right) - \frac{\sigma^2}{\sigma_a^{\,2}}\left[\cot^2\theta - \left(\frac{N^2}{\sigma^2} - 1\right)\frac{k_h^{\,2}}{k_x^{\,2}}\right] = 0. \tag{4.32}$$

The group velocity deduced from this equation is,

$$v_{gz} = \frac{\sigma(k_z + k_x\cot\theta)}{k_x^{\,2}}\left(\frac{\sigma_a}{\sigma}\right)^2 \tag{4.33}$$

But for σ/σ_a small,

$$\frac{k_z}{k_x} + \cot\theta = \pm\frac{\sigma}{\sigma_a}\left[\frac{\cot^2\theta - \left(\frac{N^2}{\sigma^2} - 1\right)\frac{k_h^{\,2}}{k_x^{\,2}}}{\cot^2\theta + \frac{k_h^{\,2}}{k_x^{\,2}}}\right]^{\frac{1}{2}} \tag{4.34}$$

then

$$v_{gz} = \pm c_a\left[\frac{\cot^2\theta - \left(\frac{N^2}{2} - 1\right)\frac{k_h^{\,2}}{k_x^{\,2}}}{\cot^2\theta + \frac{k_h^{\,2}}{k_x^{\,2}}}\right]^{\frac{1}{2}}. \tag{4.35}$$

The group velocity is real provided $\cot^2\theta > \left(\frac{N^2}{\sigma^2} - 1\right)\frac{k_h^{\,2}}{k_x^{\,2}}$. It is of the order c_a but also depends upon the Brunt frequency; it has the limiting form of neither a magnetic nor a gravity wave.

The condition for a real group velocity at low densities depends both on the wave parameters of the gravity wave at high densities and its orientation with respect to the magnetic meridian. A critical angle θ_c is implied. If the limiting gravity wave is initially propagating upwards at an angle greater than θ_c, then this wave may penetrate to all levels within the fluid. The physical attributes of the wave at low densities is not clear.

CHAPTER 5

RAY TRACING BY THE EIKONAL METHOD

Ray tracing by the Eikonal method has been well discussed by Weinberg (1962) and this will be used as the basis of the method used in this chapter. We first summarize this paper. Consider n homogeneous coupled partial differential equations which may be written in the form,

$$M(\underline{x}, i\nabla)\psi(\underline{x}) = 0, \tag{5.1}$$

where M, an $n \times n$ matrix, is characterized by the undisturbed medium and the field ψ characterizes the small perturbations. If we assume that M depends weakly on the position vector $\underline{x}$ and that all spatial derivatives of ψ contain a common factor $\exp[iT(x)]$ then, to the zero order eikonal approximation,

$$\underline{k}(\underline{x}) = \nabla T(\underline{x}). \tag{5.2}$$

For ψ to be non-vanishing requires that the determinant,

$$M(\underline{x}, \underline{k}) = 0. \tag{5.3}$$

This in turn implies a restriction on the wave number $\underline{k}$ such that,

$$G(\underline{x}, \underline{k}) = 0. \tag{5.4}$$

Here we may identify G with the dispersion relationship. Equation (5.2) however has placed a restriction on k, it must be irrotational.

To construct $\underline{k}$, we introduce a family of ray paths through the phase space defined by,

$$\underline{x} = \underline{x}(\tau), \qquad \underline{k} = \underline{k}(\tau). \tag{5.5}$$

Here τ is a convenient variable which parameterizes the path. Now along any path G will be constant, whence

$$\frac{dG}{d\tau} = 0 = \frac{\partial G}{\partial x_i} \cdot \frac{dx_i}{d\tau} + \frac{\partial G}{\partial k_i} \cdot \frac{dk_i}{d\tau} + \frac{\partial G}{\partial t} \cdot \frac{dt}{d\tau} + \frac{\partial G}{\partial \sigma} \cdot \frac{d\sigma}{d\tau} \tag{5.6}$$

and this condition is identically satisfied if we set,

$$\frac{dx_i}{d\tau} = \frac{\partial G}{\partial k_i} \qquad \frac{dk_i}{d\tau} = -\frac{\partial G}{\partial x_i}$$

$$\frac{dt}{d\tau} = -\frac{\partial G}{\partial \sigma} \qquad \frac{d\sigma}{d\tau} = -\frac{\partial G}{\partial t}\,. \tag{5.7}$$

Transforming the independent variable by using the relation

$$\frac{d}{dt} = \frac{d}{d\tau} \Big/ \frac{dt}{d\tau} \tag{5.8}$$

we derive a set of differential equations, explicit in time, which may be integrated to give the position and wave number of a wave packet as it moves through phase space.

We note the following restrictions on the method.

(a) $\underline{k}$ is irrotational; this condition is automatically satisfied in a horizontally stratified medium where k_z alone varies with height.

(b) G is real and hence M is of Hermitian form. This condition ensures that both the group velocity and $\underline{k}$ remain real. The components of ψ are thus of necessity field variables. Only in the Boussinesq approximation is this condition automatically satisfied. As a corollary we note that equation (5.2) implies a Boussinesq approximation if ψ is expressed in other than field variables since the operators contained in M are commutative only to this order of approximation.

Applying these expressions to the dispersion equation given by equation (4.4) then differential equations for the position of the packet result. Expressions for the rate of change of position are identical to those given by equations (4.13) to (4.15); consequently

$$\frac{d\underline{x}}{dt} = \underline{V}_g \tag{5.9}$$

as it should be. Additional information is obtained from the expression for the rate of change of wave number moving with a wave packet.

$$\frac{d\underline{k}}{dt} = \left(0,\ 0,\ -\ \frac{\sigma}{2}\,\frac{{\sigma_a}^2}{\sigma^2}\cdot\frac{{k_s}^2}{{k_x}^2}\right) \tag{5.10}$$

Only the vertical component of wave number varies and this variation is always of one sign. Since the differential equation describing the motion is of even order in time and hence σ we may choose σ as being of either sign. Ray reciprocity is preserved and we may traverse a given ray in either direction. Further, these expressions are parametric in σt while all wave numbers have been referred to the scale height. The equations are thus dimensionless. Equations (5.9) and (5.10) are now a consistent and soluble set of first order coupled equations amenable to numerical calculations Consider some examples.

Figure (5.1) shows the calculated ray path for a wave with parameters $k_x = 4.25$, $k_y = 0$ and $N/\sigma = 1.2$. The vertical scale of this plot is equivalent to the reduced height and is parameterized by log (σ_a^2/σ^2). The horizontal scale is similarly normalized. In the upper curve the magnetic field is directed vertically, while the ray is initialized as an upgoing gravity wave at the point X = 0.0, Z = -4.0. The circles on the ray path represent the position of a wave packet at time intervals of twenty π. The ray path is seen to refract away from the vertical and finally asymptotic to the direction of the magnetic field line in the downwards direction. Values of k_z along the trajectory agree with those found from the dispersion equation to within 0.01 percent. The ray tracing has given reliable results which can be used with some confidence. It has shown that to the order of the Boussinesq approximation, an upgoing gravity wave is converted to a downgoing Alfvén wave and that transition has occurred within a finite flight time and at some definable level in the fluid.

The next example, consider rays initially corresponding to upgoing gravity waves. Values of k_x and N/σ are unchanged but the magnetic field is now inclined at an angle of $\pm 60^\circ$ to the horizontal. This inclination of magnetic field corresponds to a continuous plot of k_z versus z as shown in Figure (4.1) and the ray trace shows similar continuous behaviour. The ray with the smaller component of wave vector

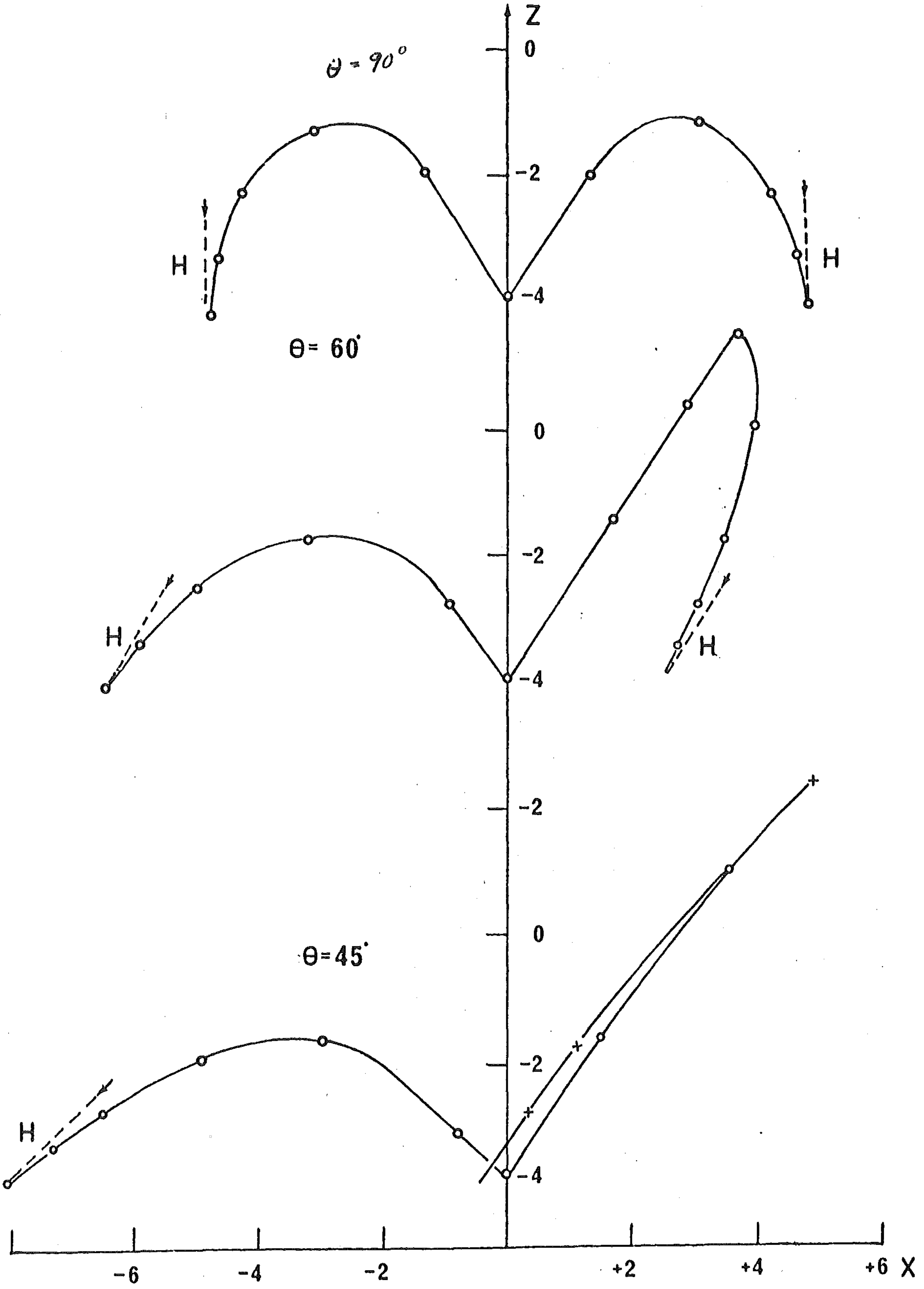

FIGURE 5.1 This plots the ray paths in normalized coordinates for waves with $k_x = 4.24$, $k_y = 0.0$ and $N/\sigma = 1.2$. Three angles of magnetic inclination are plotted. Traces are initialized as upgoing gravity waves at the point $X = 0.0$, $Z = -4.0$. The circles on the ray path represent the position of the wave packet at time intervals of twenty π.

in the direction of the magnetic field line penetrates to a greater height. This is to be expected since the ambient magnetic field has no effect when motion is directed along it. As with vertical magnetic field, the path approaches the direction of the magnetic field line in the Alfvén mode asymptote. Behaviour is thus a continuous extension to that found for a vertically directed magnetic field. The ray path for this case is plotted in Figure (5.1).

The third example, plotted in Figure (5.1), illustrates the ray path traced by a packet when wave parameters are such that one branch of the k_z versus z curve in Figure (4.1) is continuous while the other branch extends to infinity and has a cusp at this point, ($\theta = 45^o$). In the continuous left hand branch of Figure (4.1) the group velocity and hence the fluid motion is directed predominantly across the magnetic field lines. The corresponding ray traces a shallow trajectory and represents a continuous extension of the behaviour demonstrated in the previous examples. On the other hand, when motion is more along the field lines the k_z versus z plot extends to infinity and the ray path has similar behaviour. The circles represent the position of the wave packet at time intervals of twenty π ; these circles are replaced by crosses on the downwards leg. The ray trace suggests that the group velocity becomes infinite as z tends to infinity. This confirms the similar group velocity suggested in Chapter 4.

Should partial reflection be present then ray tracing will not give information of its presence. Because of this and because in some circumstances the assumptions fail, we require better solutions. There remains however the strong possibility that energy present in the form of gravity waves is converted to energy propagating in what can more accurately be described as an Alfvén mode. We therefore seek to establish the necessary conditions required for this conversion by studying the wave solutions.

CHAPTER 6

INTEGRATION OF THE WAVE EQUATION

We seek clarification of the behaviour of these waves; we seek solution to equation (4.2). Since equation (4.2) is not a standard form, numerical methods must eventually be used but first we enquire into the behaviour of the solutions.

The behaviour of the solutions to equation (4.2) when the density is large and the Alfvén velocity is small is easily found. Rewriting equation (4.2)

$$\left(1 + \frac{c_a^{\;2}}{\sigma^2}\frac{\partial^2}{\partial S^2}\right)\nabla^2 w - \frac{\partial w}{\partial z} + k_n^{\;2}\,\frac{N^2}{\sigma^2}\,w = 0. \qquad (6.1)$$

Neglecting terms of the order of $c_a^{\;2}$,

$$\frac{d^2 w}{dz^2} - \frac{dw}{dz} + k_n^{\;2}\left(\frac{N^2}{\sigma^2} - 1\right)w = 0. \qquad (6.2)$$

This equation has a solution

$$w = G_D \exp\left[ik_n\left(\frac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}} z\right] + G_U \exp\left[-ik_n\left(\frac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}} z\right] \qquad (6.3)$$

where G_D and G_U are undetermined integration constants, which can be identified with the amplitudes of downgoing and upgoing gravity waves. The solutions describes an undetermined mixture of gravity waves propagating in either direction along the z axis. These waves are unaffected by the presence of the magnetic field. Alternatively if

$$\frac{\partial^2 w}{\partial z^2} >> \frac{\partial w}{\partial z} >> w \tag{6.4}$$

equation (6.1) is approximately

$$c_a^{\,2} \frac{d_2}{dS^2} w'' + \sigma^2 w'' = 0 \tag{6.5}$$

In a medium where $c_a^{\,2}$ varies exponentially with height, the solution to this equation is

$$w'' = p_2 H_0^{(1)} (\frac{\sigma}{c_a}) + q_2 H_0^{(2)} (\frac{\sigma}{c_a}) \tag{6.6}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions of the first and second kind and of zero order. The behaviour for σ/c_a large is

$$w'' = M_U \left(\frac{c_a}{\sigma}\right)^{\frac{1}{2}} e^{\frac{i\sigma s}{c_a}} + M_D \left(\frac{c_a}{\sigma}\right)^{\frac{1}{2}} e^{\frac{-i\sigma s}{c_a}} \tag{6.7}$$

This again is an undetermined mixture of upward and downward propagating waves. Both the amplitude and wavelength depend on the wave frequency and the Alfvén velocity; they increase with height. The wave is magnetic in character.

The behaviour of the two independent solutions contained in both equation (6.3) and equation (6.7) corresponds to the behaviour associated with the four branches of the dispersion curves plotted in Figure (4.1). A solution to equation (6.1) may be approximated by an appropriate combination of equation (6.3) and equation (6.7). The integration constants however must be chosen so that the solution is physically realistic. At present we have no basis on which to choose these constants. To this end we find the behaviour of the solution to equation (6.1) at small densities.

In a stratified medium where the density varies exponentially with height, equation (4.2) may be transformed to an equation in a new independent variable depending only on the density of the medium. Put

$$\frac{\rho_0}{\rho_{00}} = e^{-z} = \lambda. \tag{6.8}$$

The reference level defined by $z = 0$ is at a density ρ_{00} so far undefined. Now

$$\frac{\sigma_a^{\,2}}{\sigma^2} = \frac{\mu H^2 k_x^{\,2}}{\rho_{00}\sigma^2} \cdot \frac{1}{\lambda} . \tag{6.9}$$

The level $z = 0$ may be defined at that level where

$$\frac{\mu H^2 k_x^{\,2}}{\rho_{00}\sigma^2} = \frac{c_z^{\,2} k_x^{\,2}}{\sigma^2} = 1. \tag{6.10}$$

Equation (4.2) becomes

$$\left[k_x^{\,2} + \frac{1}{\lambda}(ik_x\cos\theta - \sin\theta\lambda\frac{d}{d\lambda})^2\right]\left[\lambda^2\frac{d_2 w}{d\lambda^2} + \lambda\frac{dw}{d\lambda} - k_h^{\,2} w\right]$$

$$+ k_x^{\,2}\,\lambda\frac{dw}{d\lambda} + k_x^{\,2}\,k_h^{\,2}\,\frac{N^2}{\sigma^2}w = 0. \tag{6.11}$$

This is an equation of some complexity, however it is of the Fuchsian form with regular singularities only at $\lambda = 0$. Conditions are satisfied for a uniformly convergent expansion as a power series in λ (Poole, 1960, p.75). Put

$$w = \sum_{n=0}^{\infty} a_n \lambda^{\nu+n}, \tag{6.12}$$

a_n and ν are such that the series satisfies equation (4.2). Note the simplifications,

$$\left(\lambda^2 \frac{d_2}{d\lambda^2} + \lambda \frac{d}{d\lambda} - k_h{}^2\right)\lambda^{\nu+n} = [(\nu+n)^2 - k_h{}^2]\lambda^{\nu+n}, \tag{6.13}$$

and

$$\left(ik_x\cot\theta - \sin\theta\cdot\lambda\frac{d}{d\lambda}\right)^2\lambda^{\nu+n} = [(\nu+n)\sin\theta - ik_x\cos\theta]^2\lambda^{\nu+n} \tag{6.14}$$

Thus,

$$\sum_{n=0}^{\infty} [(\nu+n)\sin\theta - ik_x\cos\theta]^2[(\nu+n)^2-k_h{}^2]\ a_n\ \lambda^{\nu+n-1}$$
$$+ \sum_{n=0}^{\infty} k_x{}^2[(\nu+n)^2 + (\nu+n) + k_h{}^2(\ell^2-1)]\ a_n\ \lambda^{\nu+n} = 0, \tag{6.15}$$

The coefficient of the lowest power in λ gives the indicial equation which must be zero if equation (6.13) is to be satisfied.

$$(\nu\ \sin\theta - ik_x\cos\theta)^2(\nu^2-k_h{}^2) = 0, \tag{6.16}$$

whence

$$\nu = k_h,\ -k_h,\ ik_x\cot\theta,\ ik_x\cot\theta. \tag{6.17}$$

Near $\lambda = 0$ there are four independent solutions corresponding to the four indicial roots having the behaviour

$$w \approx \lambda^{k_h},\ \lambda^{-k_h},\ \lambda^{+ik_x\cot\theta},\ \log\lambda\cdot\lambda^{ik_x\cot\theta} \tag{6.18}$$

or in terms of the vertical scale z

$$w \approx e^{-k_h z}, e^{+k_h z}, e^{-ik_x \cot\theta . z}, -ze^{-ik_x \cot\theta . z} \tag{6.19}$$

Two of these approximations represent evanescent waves while the remaining two are propagating. Each of the pairs contains an approximation in which w diverges as z tends to infinity.

Equating the coefficients of $\lambda^{\nu+n}$ in equation (6.15) then a recurrence relationship between the undetermined coefficients of the power series is found,

$$[(\nu+n+1)\sin\theta - ik_x\cos\theta]^2[(\nu+n+1)^2 - k_h{}^2]a_{n+1}$$
$$+ k_x{}^2[(\nu+n)^2 + (\nu+n) + k_h{}^2(\;{}^2-1)]a_n = 0$$
$$(6.20)$$

The series solution is thus

$$w = a_0\lambda^\nu \left\{1 + \sum_{n=1}^{\infty} \frac{(-1)^n k_n^{2n} \prod_{r=0}^{n-1}[(\nu+r)^2+(\nu+r)+k_h{}^2(\ell^2-1)]}{\prod_{r=1}^{n}[(\nu+r)\sin\theta - ik_x\cos\theta]^2[(\nu+r)^2-k_h{}^2]}\lambda^n\right\} \tag{6.21}$$

The series expansion for the repeated root may also be found from this series expansion and may be formally written as

$$w = \log\lambda \sum_{n=0}^{\infty} a_n\lambda^{\nu+n} + \sum_{n=1}^{\infty} a_n'\lambda^{\nu+n},$$

$$a_n' = \frac{\partial a_n}{\partial\nu}. \tag{6.22}$$

The series are complicated, there is little hope of direct summation or even asymptotic approximations for large λ. By virtue of the double poles in the series given by equation (6.22), a representation as a contour integral may be found. This approach does not yield information of much value. Now we examine the solutions near $\lambda = 0$ to see how the energy flux behaves.

Using the expressions for the energy flux vector given in Chapter (3), then the vertical component of this vector vanishes near $\lambda = 0$ for those solutions designated by the indicial roots $\nu = \pm k_h$. This is as expected for evanescent waves. The behaviour of the solutions for the repeated root $\nu = ik_x \cot\theta$ at large z is,

$$w = \mathcal{R}e \exp(k_x \cdot x + k_y \cdot y - ik_x \cot\theta \cdot z - \sigma t) \tag{6.23}$$

$$w = \mathcal{R}e\, z \exp(k_x \cdot x + k_y \cdot y - ik_x \cot\theta \cdot z - \sigma t) \tag{6.24}$$

Both represent plane waves with a wave vector satisfying,

$$\underline{k} \cdot \underline{H} = 0 \tag{6.25}$$

But the fluid motion also satisfies $\nabla . \underline{u} = 0$, whence

$$\underline{k} \cdot \underline{u} = 0 \tag{6.26}$$

For $k_y = 0$, the vectors $\underline{u}$ and $\underline{H}$ are therefore parallel vectors. The solution given by equation (6.23) admits the interpretation where the fluid moves only along the magnetic field line, the amplitude of this motion is constant. Magnetic perturbations are zero since in this limit $\underline{u} \times \underline{H} = 0$. The second solution given by equation (6.24) represents a similar motion, but one in which the ampltiude of the oscillation increases linearly in the vertical direction. This solution is best interpreted by applying the frozen field theorem. Consider a column of fluid contained within a cylinder aligned along the direction of the magnetic field. By the frozen field theorem, the cross-section of this tube is such that

$$\int \underline{H}\, d\underline{a} = \text{Constant} \tag{6.27}$$

If the cross-section of this tube is allowed to vary sinusoidally with an amplitude independent of the distance along the tube, then the height of a given parcel of fluid above some fixed level will also vary sinusoidally but with an amplitude proportional to the distance measured along the tube. In this way the total volume below this fluid element is kept constant. It will be noted that neither of these asymptotic solutions transmits energy. The second solution however represents a fluid motion 'imposed' by perturbations in the magnetic field while the first solution represents a possible motion without these imposed variations.

We now show that only two solutions are sufficient to uniquely determine the nature of the waves propagating at large values of λ. Let S_1 and S_2 represent these wave solutions which at large values of λ decompose in the following way

$$S_1 = M_{U1} + M_{D1} + G_{U1} + G_{D1},$$

and

$$S_2 = M_{U2} + M_{D2} + G_{U2} + G_{D2}, \tag{6.28}$$

where M_{U1} represents the upgoing magnetic wave in the first solution and other terms are interpreted in a corresponding way. Now in general

$$\frac{M_{U1}}{M_{U2}} \neq \frac{M_{D1}}{M_{D2}}, \tag{6.19}$$

and

$$\frac{G_{U1}}{G_{U2}} \neq \frac{G_{D1}}{G_{D2}} \tag{6.30}$$

since S_1 and S_2 are independent solutions. It is therefore possible to eliminate any one mode from the given equations and so find a unique relationship between the remaining three modes. If we choose to eliminate the upgoing magnetic mode then a relationship exists between the upward propagating gravity mode and downwards propagation in both the gravity and magnetic modes. Furthermore the proportion of the energy propagated downwards in each mode is known. If S_1 and S_2 are solutions with a non-radiative upper boundary the derived solution is energetically closed and unique.

Of the four possible solutions given by the four indicial roots two only are needed. The solutions designated by the indicial roots $\nu = k_h$ and $\nu = ik_x \cot \theta$ have bounded energy densities near $\lambda = 0$ and these are to be preferred to the other two solutions in which the energy densities diverge at small values of λ. These two solutions represent an equilibrium wave solution with an energy minimum; this is considered to be the more probable equilibrium state to which the medium will tend if initially unperturbed.

Some simple transformations may be applied when $\theta \neq 0$. Equation (6.1) can be written as

$$\frac{c_a^{\,2}\sin^2\theta}{\sigma^2}\left(\frac{d}{dz} + ik_x\cot\theta\right)^2\left(\frac{d_2w}{dz^2} - k_h^{\,2}w\right) + \frac{d_2w}{dz^2} - \frac{dw}{dz} + k_h^{\,2}\left(\frac{N^2}{\sigma} - 1\right)w$$

$$= 0. \qquad (6.31)$$

But, sin θ may be associated with either ρ_0 or H since

$$\frac{c_a^{\,2}\sin^2\theta}{\sigma^2} = \frac{\mu H^2}{\left(\frac{\rho_0}{\sin^2\theta}\right)\sigma^2} = \frac{\mu(H\sin\theta)^2}{\rho_0\sigma^2}, \qquad (6.32)$$

and represents either a modification to the density of the medium or else the inclusion of only the vertical component of the magnetic field. Further, when $\theta = 90^{\circ}$ the imaginary term contained within the first bracket vanishes as it also vanishes when $k_x = 0$. Solutions to equation (6.31) found when $\theta = 90^{\circ}$ also apply when $k_x = 0$ provided a correction is made either to the magnetic field strength or the density of the medium. Putting,

$$k_x = k_h\cos\phi, \qquad (6.33)$$

where θ is the angle between $\underline{k}_h$ and the magnetic meridian, then equation (6.32) becomes,

$$\frac{c_a^{\,2}\sin^2\theta}{\sigma^2}\left(\frac{d}{dz} + ik_h\cos\phi\cot\theta\right)^2\left(\frac{d_2w}{dz^2} + k_h^{\,2}w\right)$$

$$+ \frac{d_2w}{dz^2} - \frac{dw}{dz} + k_h^{\,2}\left(\frac{N^2}{\sigma^2} - 1\right)w = 0 \qquad (6.34)$$

Further write

$$\cot \psi = \cos \phi \cot \theta \tag{6.35}$$

Now define the level z = 0 where,

$$\frac{c_a^{\,2} k_h^{\,2}}{\sigma^2} = 1 \tag{6.36}$$

Equation (6.45) then becomes

$$\sin^2\psi \; e^{-\left(z - \log \frac{\cos^2\phi}{\cos^2\phi + \sin^2\psi}\right)} \left(\frac{d}{dz} + ik_h \cot\psi\right)^2 \left(\frac{d^2w}{dz^2} - k_h^{\,2} w\right)$$

$$+ \frac{d^2w}{dz^2} - \frac{dw}{dz} + k_h^{\,2}\left(\frac{N^2}{\sigma^2} - 1\right) w = 0 \tag{6.37}$$

In general, a transformation on both θ and z reduces the equation to an equivalent form where $k_y = 0$. It is sufficient to find the behaviour only when $k_y = 0$.

CHAPTER 7

NUMERICAL SOLUTIONS

In this chapter we integrate the differential equation describing the motion numerically. The form used here is derived from equation (6.1) by putting $k_y = 0$ and defining the level $z = 0$, where $c_a^2 k_x^2/\sigma^2 = 1$. Thus,

$$e^{+z}\left(\sin\theta \frac{d}{dz} + ik_x\cos\theta\right)^2 \left(\frac{d^2w}{dz^2} - k_x^2 w\right)$$
$$+ k_x^2 \left[\frac{d^2w}{dz^2} - \frac{dw}{dz} + k_x^2\left(\frac{N^2}{\sigma^2} - 1\right)w\right] = 0 \qquad (7.1)$$

Define S_1 to be the solution found when $\nu = ik_x\cot\theta$ and S_2 for $\nu = k_x$. For the reasons given in the previous chapter these are the only solutions considered.

Solutions S_1 and S_2 are found by the following numerical methods.

(1) w and its derivatives are found from the series expansion in λ given by equation (6.21). The series is used no further than $\lambda = 1$ due to poor convergence.

(2) The values so found are used as boundary conditions to further extend the solutions by the numerical integration of the differential equation (7.1). Details of this are found in Appendix IV.

(3) The accuracy of the numerics is checked by computing the energy flux as integration proceeds. The integrity of the calculations is judged by the conservation of energy.

(4) The solutions so found are interpreted in terms of waves satisfying the WKB approximation at large values of -z.

WKB approximations are derived as approximate solutions to a second order wave equation, they do not strictly apply to an equation of the fourth order. However, we use these as a basis for wave decomposition. If k(z) is a slowly varying function of z, then approximations to the wave equation,

$$\frac{d_2 w}{dz^2} + k^2(z)w = 0 \tag{7.2}$$

are,

$$w_1 = e^{\pm i\phi_0}. \tag{7.3}$$

Or to a higher order

$$w_2 = \frac{1}{k^{\frac{1}{2}}} e^{\pm i\phi_0}. \tag{7.4}$$

The phase integral ϕ_0 is defined as

$$\phi_0 = \int_{z_0}^{z} k(z)\cdot dz . \tag{7.5}$$

The accuracy of these approximations to an equation of a higher order than equation (7.2) can only be judged by the size of terms neglected when w_1 or w_2 is substituted into the equation. w_2 neglects smaller terms than w_1 when substituted into equation (7.1) and this is considered the better approximation.

Since in equation (7.5) $\phi_0(z_0) = 0$, the phase integral contains an arbitrary zero. The phase of a wave can be measured as the phase displacement ϕ from the phase integral ϕ_0.

ϕ is also measured with respect to an arbitrary zero and has no absolute meaning. The constancy of ϕ however, gives an indication of the goodness of fit of ϕ_0 to the wave mode and an indirect check of the accuracy of the approximations made. Here we will define $\phi_0 = 0$ at $z_0 = 0$.

Decomposition of solutions into wave components is made by the following method. At sufficiently large negative values of z, either solution S_1 or S_2 may be approximated by the WKB expression

$$S = \alpha M_d + \beta G_u + \gamma G_d + \delta M_u \tag{7.6}$$

where

$$\alpha = \frac{1}{[\mathrm{Re}(k_z)]^{\frac{1}{2}}} \exp [i\phi_0(z)] \quad \text{etc.} \tag{7.7}$$

Here $\mathrm{Re}(k_z)$ is the real part of the complex wave number and ϕ_0 is the complex phase integral of the corresponding mode. The symbol M refers to a magnetic mode while G refers to a gravity mode. The suffix d indicates a wave with a downward propagation of energy and u denotes upwards propagation. Here G_u corresponds to a gravity wave with an upwards energy flux and hence a negative vertical component of wave number. Multipliers M_u, M_d, G_u, G_d are complex numbers which can be interpreted as follows; $|M_u|$ is the amplitude of the upgoing magnetic mode and $\mathrm{Arg}(M_u)$ is the phase displacement ϕ measured with respect to the phase integral $\mathrm{Re}(\phi_0)$. The other symbols are defined in a similar manner. Choosing four height levèls, then applying equation (7.6) at each level gives a soluble set of simultaneous equations from which the amplitudes and phase displacements can be estimated. The function S and its first three derivatives could also be used to form a soluble simultaneous set. However, the amplitude of the derivatives of S increases with the order of differentiation; an ill-conditioned set with poor arithmetic accuracy results.

We wish to test the hypothesis of mode conversion; assume that the multipliers M_u etc. have been determined. Write

$$|M_u| = m_u$$

$$\mathrm{Arg}(M_u) = \phi_{m_u} \tag{7.8}$$

with other quantities similarly defined. A linear combination of the independent solutions S_1 and S_2 can be formed in the proportion of p to q. p and q may be complex multipliers. Thus

$$\begin{aligned} S &= pS_1 + qS_2 \\ &= (pm_{d1}e^{i\phi_{m_{d1}}} + qm_{d2}e^{i\phi_{m_{d2}}})\alpha + (pg_{u1}e^{i\phi_{g_{u1}}} + qg_{u2}e^{i\phi_{g_{u2}}})\beta \\ &+ (pg_{d1}e^{i\phi_{g_{d1}}} + qg_{d2}e^{i\phi_{g_{d2}}})\gamma + (pm_{u1}e^{i\phi_{m_{u1}}} + qm_{u2}e^{i\phi_{m_{u2}}})\gamma \end{aligned} \tag{7.9}$$

S is also a solution of equation (7.1). We now choose to make to coefficient of the phase integral δ zero and so construct a solution in which the upwards propagation of energy is allowed in only the gravity mode. Complete mode conversion requires that the coefficient of γ also vanishes.

$$pg_{d1}e^{i\phi_{g_{d2}}} + qg_{d2}e^{i\phi_{g_{d2}}} = 0 \qquad (7.10)$$

$$pm_{d1}e^{i\phi_{m_{d1}}} + qm_{d2}e^{i\phi_{m_{d2}}} = 0 \qquad (7.11)$$

Comparing the coefficients of p and q then

$$\frac{m_{u1}}{m_{u2}} = \frac{g_{d1}}{g_{d2}} \qquad (7.12)$$

$$\phi_{m_{u1}} - \phi_{m_{u2}} = \phi_{g_{d1}} - \phi_{g_{d2}} \qquad (7.13)$$

Similar expression can be found to verify the converse proposition, that an up-going magnetic wave is converted to a down-going gravity wave. However, the differential equation contains only σ^2 and it is unchanged when σ is replaced by $-\sigma$. Verification of the converse proposition is therefore implied.

Three cases are now considered; that when the ambient magnetic field is vertical ($\theta = 90^{\circ}$), the inclined magnetic field, and when the magnetic field is horizontal ($\theta = 0^{\circ}$). The conversion of a gravity wave into a magnetic wave will be demonstrated in each case.

The Magnetic Field Vertical

Figure (7.1) plots solutions for $\theta = 90^{\circ}$, $k_x = 4.25$, $N/\sigma = 1.2$. For $\theta = 90^{\circ}$, w is always real since the imaginary terms have vanished from equation (7.1). The plot shows that w is composed of two sinusoids, the one of longer vertical wave length approximating that of a gravity wave, the other showing an increase of wave length with height characteristic of an Alfvén mode.

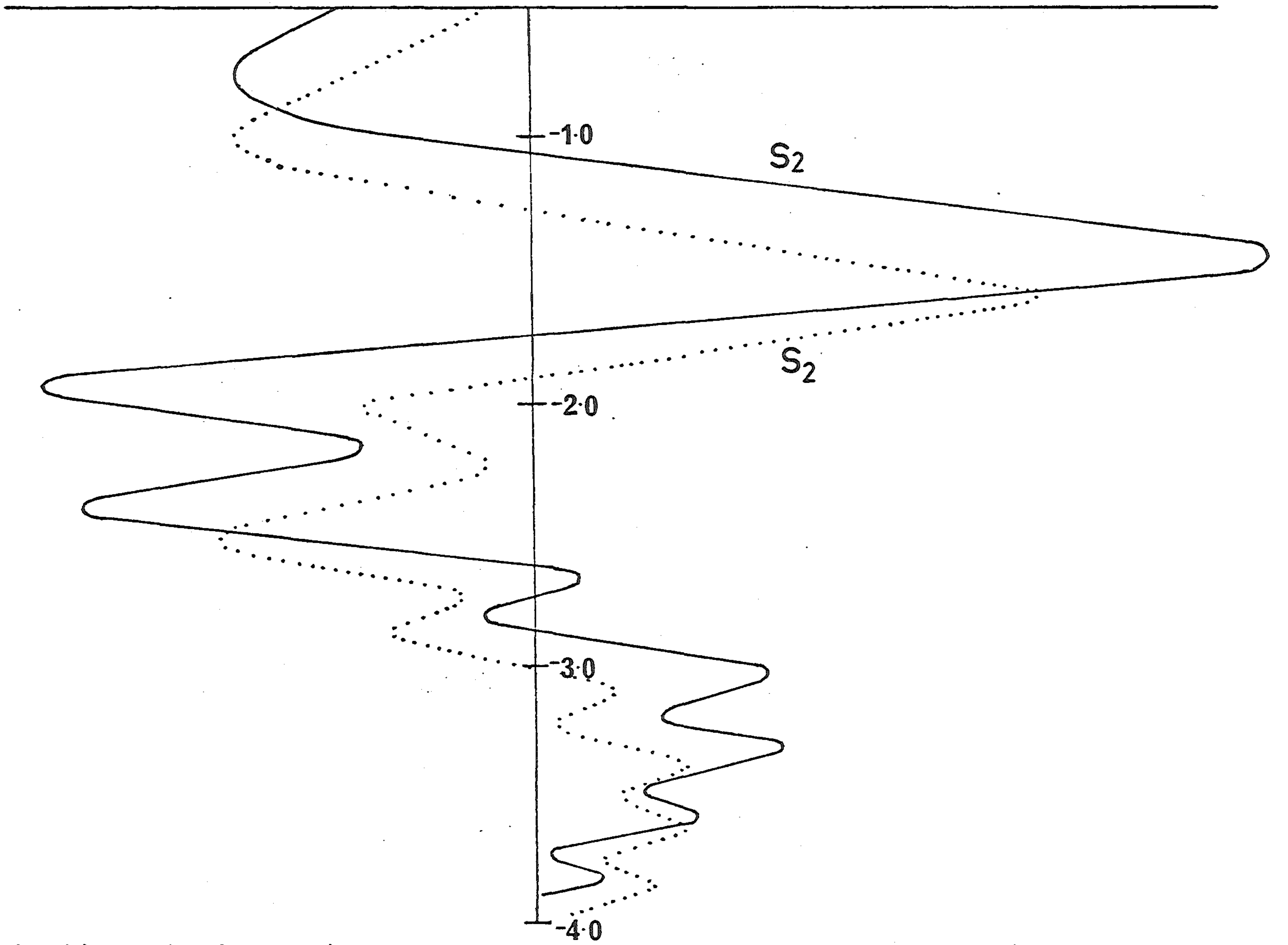

FIGURE 7.1 This graphs the solutions S_1 and S_2 against the reduced height. Perturbations are plotted to an arbitrary scale suitably separated to avoid confusion. Wave parameters are, $\theta = 90^o$, $k_x = 4.25$, $N/\sigma = 1.2$.

Table (7.1) gives some results of calculations and tests the conditions given by equations (7.12) and (7.13). The phase relationships are expressed as phase differences between the solutions S_1 and S_2. The phase differences are indeed constants. The values are centred on 45°, variations from this value are small and may be attributed to arithmetic noise. The phase condition for complete mode conversion is satisfied. The fourth column of Table (7.1) shows that the amplitude condition is also satisfied to a good order of precision save near the level where mode conversion takes place. We conclude that, for at least this particular case, complete mode conversion is a valid physical process to within one percent accuracy. Also that the lower order approximations have given a true indication that this process does indeed occur.

The solution representing an upward flow of energy in the gravity mode only is now constructed. Imperfections in the calculation give rise to a vertical component of energy flux, which when compared with the upwards gravity flux estimates the overall reliability of the computations. These ratios, tabulated in column five, are small and the fidelity of the solution is assured.

An immediate interpretation of these solutions is that they represent two independent "standing" modes which are composed of a mixture of both gravity and magnetic waves; the vertical component of the energy flux is zero while the horizontal components are bounded. For these "standing" modes the lower boundary condition corresponding to some prescribed spatial variation of w at some fixed value of z is easily inserted.

Z	S_1		S_2		Phase Difference		$\lvert M/G\rvert_{S_1} / \lvert M/G\rvert_{S_2}$	$\frac{\text{TOTAL FLUX}}{\text{GRAVITY FLUX}}$ %
	|Magnetic|	|Gravity|	|Magnetic|	|Gravity|	Magnetic S_1-S_2	Gravity S_1-S_2		
-2.25	258.1	511.0	1.847×10^{-1}	3.549×10^{-1}	44.5°	45.8°	0.970	8.0×10^{-4}
-2.65	179.4	420.1	1.296×10^{-1}	3.012×10^{-1}	45.3°	44.0°	0.993	1.2×10^{-3}
-3.05	128.4	328.3	0.930×10^{-1}	2.411×10^{-1}	44.9°	44.9°	1.014	2.4×10^{-3}
-3.45	92.5	266.5	0.673×10^{-1}	1.926×10^{-1}	45.3°	45.2°	0.993	3.8×10^{-3}
-3.85	67.6	215.9	0.489×10^{-1}	1.573×10^{-1}	45.1°	44.9°	1.008	1.35

TABLE 7.1 Tabulation of the amplitudes of the magnetic and gravity components for the two independent and converging solutions S_1 and S_2 derived by application of the WKB approximation. The phase relations are measured with respect to the phase integral which is zeroed at the level $z = 0$. The phase and amplitude relationships are such that elimination of the magnetic wave also eliminates the gravity mode. Wave parameters are, $\theta = 90^{\circ}$, $k_x = 4.25$, $N/\sigma = 1.2$.

The study of "standing" modes is not our prime concern, rather the interpretation in terms of propagating waves.

Now the dispersion equation associated with the differential equation is,

$$e^{z}k_z^{\,4} + k_x^{\,2}(e^{z}-1)k_z^{\,2} - ik_x^{\,2}k_2 + k_x^{\,4}\left(\frac{N^2}{\sigma^2} - 1\right) = 0 \qquad (7.14)$$

Inspection of the computations shows that this equation has roots of the form $\pm k_m + iK$, $\pm k_g - iK$ where k_m, k_g, K are appropriate values. This proposition may be checked by direct substitution. The phase integrals are then related,

$$\begin{aligned} \delta &= \alpha^*, \\ \gamma &= \beta^*. \end{aligned} \qquad (7.15)$$

where the asterisk denotes complex conjugation. Clearly since both S_1 and S_2 are always real then

$$\begin{aligned} M_u &= M_d^*, \\ \text{and} \quad G_d &= G_u^*. \end{aligned} \qquad (7.16)$$

Both S_1 and S_2 can be interpreted as being composed of counter-flowing gravity and magnetic pairs of equal amplitude.

The relations given by equation (7.16) are sufficient to show that a gravity wave can never be completely reflected in the same mode. Suppose the coefficients of both α and δ are zero in equation (7.9). Then

$$\frac{m_{d_1}}{m_{u_1}}\, e^{i(\phi_{m_{d_1}} - \phi_{m_{u_1}})} = \frac{m_{d_2}}{m_{u_2}}\, e^{i(\phi_{m_{d_2}} - \phi_{m_{u_2}})} \qquad (7.17)$$

Equation (7.16) implies that

$$\phi_{m_{d_1}} = \phi_{m_{d_2}} \tag{7.18}$$

S_1 and S_2 therefore satisfy identical WKB approximations and are not independent, as they should be.

The Inclined Magnetic Field (Meridional Motion)

Consider now the case when the ambient magnetic field is inclined at some arbitrary angle θ to the horizontal. The expansion for the vertical component of the perturbation velocity given by equation (6.1) is now complex, while the dispersion equation corresponding to this differential equation yields four unequal roots. The gravity and magnetic limits of equation (4.2) however still have the same functional form as those found for $\theta = 90^{\circ}$. We therefore expect that each of the convergent solutions S_1 and S_2 corresponding to the indicial roots $\nu = ik_x\cot\theta$ and $\nu = k_x$ (equation 6.17) will connect to, at the most, four propagating waves associated with the four unequal roots of the dispersion equation. Two of these will be gravity modes while the remaining will be Alfvén modes. Unlike the special case for $\theta = 90^{\circ}$ when w is real, the construction of wave solutions from duct-like solutions is no longer necessary. The interpretation of the complex solutions in terms of upgoing and downward wave propagation can be found by direct application of equation (7.6) and the method outlined at the beginning of that section. Further, the integrity of the numerics may be directly checked by estimating the vertical component of the energy flux associated with each solution S_1 and S_2 independently. As before we consider the restricted case where $k_y = 0$ and confine the fluid motion to lie entirely in the meridional plane.

Figure (7.2) plots the third derivative of the solution S_1 found when $k_x = 4.25$ radians per reduced height, $\theta = 60^\circ$ and $N/\sigma = 1.2$. Since the vertical wave numbers of the magnetic modes greatly exceeds those of the gravity modes, the amplitude of the gravity component is greatly suppressed by differentiation. The wave form of the magnetic modes may thus be more clearly seen. The solid curve plots the real part while the dashed curve plots the imaginary part of d_3S_1/dz^3. Figure (7.2) has the appearance of beating waves whose wave lengths differ by only a small amount. The wave lengths decrease as the reduced height decreases as to be expected for the Alfvén modes. Figure (7.2), right hand curves, plots the comparable function S_2. Here beating is absent, the real component lags the imaginary component by approximately a quarter of a wave length. S_2 thus approximates the expected function which would represent an upward propagating Alfvén wave.

Decomposing the wave functions S_1 and S_2 into their component waves by the method outlined at the beginning of the last section yields Table (7.2). Clearly S_1 is composed of an almost equal mixture of all four modes while S_2 represents an upgoing gravity mode accompanied by a downgoing magnetic wave. In S_2, the alternate wave pair, downgoing magnetic partnered by an upgoing gravity wave, represents less than one percent of the dominant's amplitude or 0.01 percent of the transmitted energy. The fidelity of the solutions is assured by the

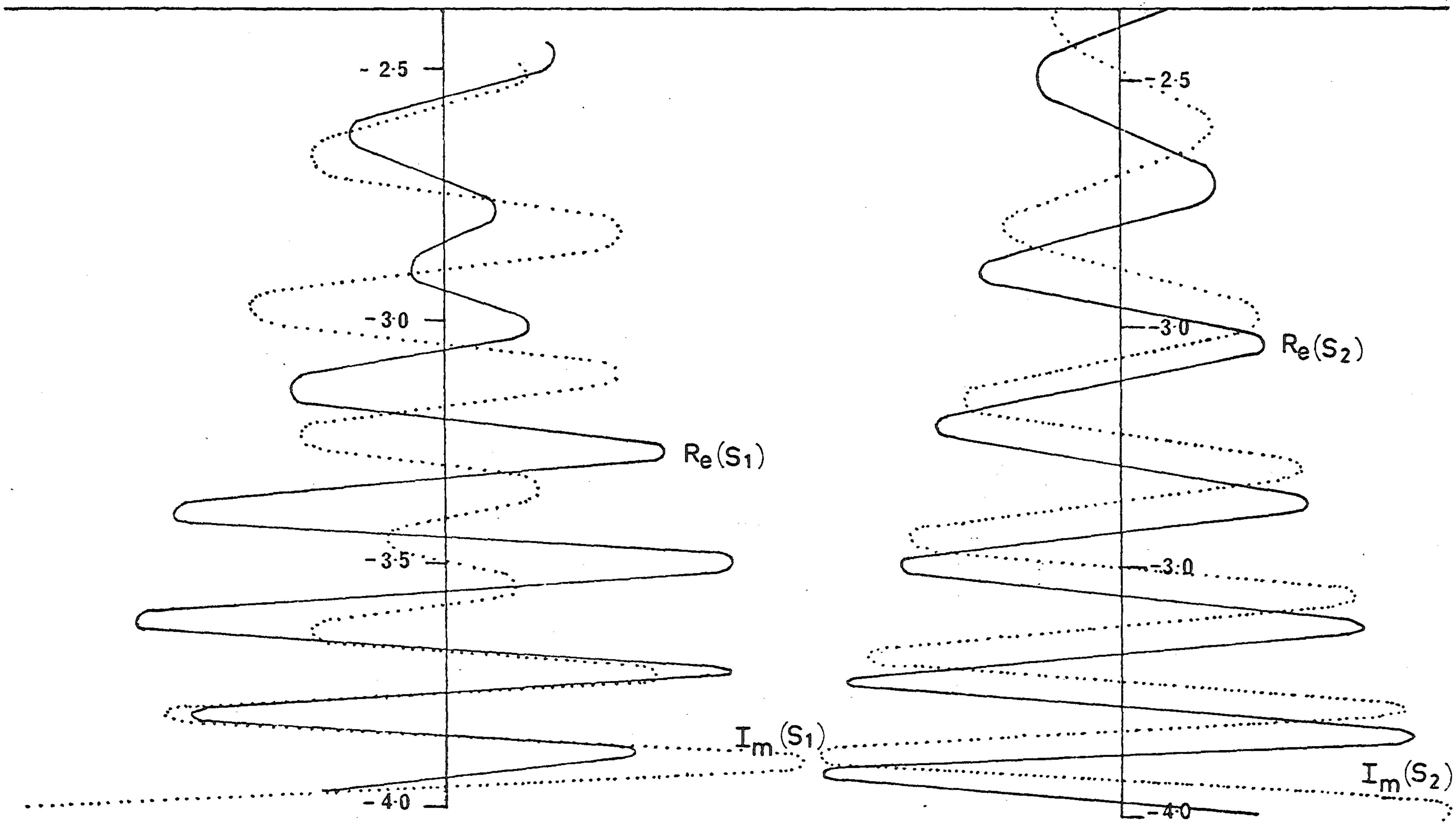

FIGURE 7.2 This shows the third derivatives of S_1 (left hand curve) and S_2 plotted to an arbitrary amplitude. The solutions have parameters $N/\sigma = 1.2$, $k_x = 4.25$ and $\theta = 60^{\circ}$. Differentiation suppresses the long wavelengths of the gravity mode and shows the compound magnetic mode contained in these solutions.

Z	Upgoing Magnetic		Downgoing Gravity		Upgoing Gravity		Downgoing Magnetic		Flux Ratio
	$\times 10^{-2}$		$\times 10^{-2}$		$\times 10^{-2}$		$\times 10^{-2}$		$\% \times 10^{-3}$
First Solution									
-2.25	10.29	241.0	17.61	15.1	15.96	6.4	6.34	150.8	1.3
-2.65	6.61	239.5	14.05	14.1	13.11	6.4	4.72	150.7	1.7
-3.05	4.53	238.7	11.17	14.3	10.78	6.8	3.51	150.5	2.8
-3.45	3.20	238.3	9.01	14.1	8.82	6.7	2.62	150.4	3.0
-3.85	2.30	238.9	7.31	14.1	7.21	6.7	1.96	151.3	3.1
Second Solution									
-2.25	18.70	46.0	32.10	180.3	0.27	96.0	0.05	328.1	0.2
-2.65	12.02	44.5	25.49	179.3	0.12	119.3	0.02	356.5	1.1
-3.05	8.24	43.9	20.34	179.4	0.05	32.2	0.01	193.5	4.8
-3.45	5.81	43.5	16.38	179.3	0.02	127.4	0.02	175.7	5.3
-3.85	4.17	43.8	13.29	179.3	0.00	46.7	0.01	45.6	6.7
Derived Solution									
			$\times 10^{-3}$				$\times 10^{-1}$		
-2.25			9.36	233.4	1.61	106.5	6.32	150.7	
-2.65			4.08	324.6	1.32	106.5	4.71	150.7	
-3.05			1.58	156.3	1.08	106.7	3.52	150.6	
-3.45			2.11	151.7	0.88	106.7	2.63	150.6	
-3.85			2.67	246.1	0.73	106.7	1.96	150.9	

TABLE 7.2 This shows the decomposition of the solutions S_1 and S_2 into gravity and magnetic wave components. The equivalent amplitudes are tabulated in the first column while the second column gives the phase displacement in degrees. The phase is measured relative to the phase integral arbitrarily zeroed at the level z = 0. The final column is the ratio of the total flux to the upgoing gravity flux. The derived solution is a linear combination of S_1 and S_2 such that the downgoing magnetic component is eliminated.

precision to which the energy flux is closed. This is shown in the final column of Table (7.2). These properties of the solution S_2 are sufficient to verify the hypothesis of complete mode conversion from gravity waves to Alfvén waves. Nevertheless we add weight to the argument by constructing a linear combination of S_1 and S_2 in which the amplitude of the upgoing Alfvén wave is made zero. Complete mode conversion requires that the gravity wave partner will show only arithmetic noise rather than a coherent wave structure; its amplitude negligible and its phase random. These characteristics are clearly demonstrated in the parameters of the derived solution shown in Table (7.2).

To facilitate the rapid numerical checking of the hypothesis of mode conversion, it is judged sufficient to merely record the amplitude ratios $|M|_{S_1}/|M|_{S_2}$ and the corresponding gravity amplitude ratio $|G|_{S_1}/|G|_{S_2}$ for those wave components with positive wave numbers. The relative phases of these waves is also recorded. Constancy of these parameters assures the joint elimination of this wave pair between the solutions S_1 and S_2. As a further check, we perform this elimination then measure the amplitude of the residual downgoing gravity wave. This amplitude, normalized by the amplitude of the retained gravity component, estimates the fraction of the gravity wave's amplitude reflected in the same mode. Table (7.3) summarizes results derived from integrations taken over the range of magnetic inclinations. The horizontal wave numbers used here do not vary, $k_x = 4.25$, $k_y = 0.0$, while the ratio of the Brunt to wave frequency is $N/\sigma = 1.2$. The mode conversion hypothesis is again supported. The amplitude ratios and the phase differences are constant well within one

Inclination	90°	85°	75°	60°	45°	30°	15°
$\lvert M\rvert_{S_2}/\lvert M\rvert_{S_1}$ $\Delta\phi$	$1.373\text{x}10^{-3}$ 45.0°	$5.028\text{x}10^{-3}$ 17.4°	0.362 72.8°	$1.817\text{x}10^{2}$ 194.5°	$2.442\text{x}10^{3}$ 13.7°	$6.739\text{x}10^{4}$ 66.3°	$1.691\text{x}10^{-1}$ 132.5°
$\lvert G\rvert_{S_2}/\lvert G\rvert_{S_1}$ $\Delta\phi$	$1.374\text{x}10^{-3}$ 45.0°	$5.022\text{x}10^{-3}$ 17.6°	0.362 72.8°	$1.821\text{x}10^{2}$ 194.9°	$2.441\text{x}10^{3}$ 13.7°	$6.763\text{x}10^{4}$ 66.3°	$1.692\text{x}10^{-1}$ 132.5°
Ratio	0.999	1.001	1.000	0.998	1.000	0.996	1.000
Residual %		0.014	0.16	0.093	0.50	0.61	0.49

TABLE 7.3 This table summarizes the relationship between estimates of amplitude and phase of the magnetic and gravity modes when k_z is positive. Values, derived by the application of the WKB approximation to the two independent solutions S_1 and S_2, show that both the magnetic and gravity components have proportional amplitudes and equal phase displacements. Elimination of the upgoing magnetic wave between S_1 and S_2 thus eliminates the downgoing gravity wave. The final row of the table gives the residual amplitude of the upgoing gravity wave expressed as a percentage of that of the downgoing gravity mode. Wave parameters here used are, $k_x = 4.25$, $k_y = 0.0$, $N/\sigma = 1.2$.

percent thus allowing the joint elimination of the wave pair with negative wave numbers. The reflection coefficient which is the square of the tabulated amplitude ratios is less than 0.01 percent. The hypothesis is supported to at least this order.

The solutions S_1 and S_2 being in general complex have decomposed into wave components by direct methods. The solution for $\theta = 90^{\circ}$ is degenerate due to the symmetry of the governing equation. We now examine the continuity of the wave decomposition near $\theta = 90^{\circ}$. Clearly analytic continuation of waves should exist near this point and we seek verification of this. Table (7.4) gives details of this continuation. The first line gives the wave amplitude normalized by the appropriate amplitude for $\theta = 90^{\circ}$ and also its relative phase. The continuation on the second line represents values computed for negative values of θ. Only values for positive k_z are tabulated and these show the required continuity across $\theta = 90^{\circ}$. The tabulations show spread since near the singularity the imaginary part of the perturbation velocity vanishes and the arithmetic accuracy correspondingly diminished. That this verification also holds for negative k_z may be shown from the relation

$$W(\theta) = W^*(-\theta) \tag{7.19}$$

a result obtained directly from equation (7.1).

We now seek exception to the behaviour already established. Yanowitch (1967) when considering the effects of viscosity on the propagation of gravity waves found that reflection dominated the absorption when the vertical wave

θ	85°		87°		89°		90°
Magnetic Wave; S_1	0.44	53.0°	0.69	39.2°	0.95	15.7°	1.00
	0.43	51.8°	0.67	38.4°	0.94	15.2°	1.00
Gravity Wave; S_1	0.44	46.2°	0.68	34.8°	0.95	13.9°	1.00
	0.44	43.4°	0.68	33.9°	0.95	14.1°	1.00
Magnetic Wave; S_2	3.08	-9.4°	1.99	-5.5°	1.03	-1.6°	1.00
	0.28	-10.3°	0.48	-5.8°	0.78	-1.5°	1.00
Gravity Wave; S_2	2.98	-16.3°	1.94	-10.2°	1.24	-3.6°	1.00
	0.29	-19.5°	0.48	-11.1°	0.78	-3.5°	1.00

TABLE 7.4 The variation of amplitude and phase near $\theta = 90^{\circ}$, for waves with parameters of $N/\sigma = 1.2$, $k_x = 4.25$, $k_y = 0.0$. Values are measured relative to those found for $\theta = 90^{\circ}$. Solutions are thus continuous over $\theta = 90^{\circ}$.

length of the gravity waves became long compared to the scale of variation of the viscous effects. The present wave equation shows some similarities with that studied by Yanowitch; viscosity becomes equivalent to an complex Alfvén velocity. It seems reasonable to examine the long vertical wave lengths. This is not without difficulty since here the WKB approximations lose validity and we can expect little more than a check on the inferences already made.

Table (7.5) gives estimates of the fractional reflection suffered by an upgoing gravity wave. This is calculated by linearly combining the solutions S_1 and S_2 so as to eliminate the upgoing magnetic wave. Thus energy is propagated upwards only in the gravity mode and reflected as a gravity wave or converted into energy resident in the Alfvén mode. The ratio $|G_d|^2/|G_u|^2$ estimates the proportion of the energy reflected in the same mode. The ratio is tabulated as a function of the asymptotic vertical wave number of the gravity wave given by the relation

$$\beta^2 = k_x^{\ 2} \left(\frac{N^2}{\sigma^2} - 1\right) - \tfrac{1}{4} \qquad (7.20)$$

The reflection coefficient varies rapidly with β, even more steeply than that found by Yanowitch for reflection in a viscous medium. However, as previously noted, no great reliance can be placed on these values. They do indicate that reflection without mode conversion is possible only in the limit of infinite vertical wave length.

β	0.0	0.1	0.170	0.316
$\lvert G_D\rvert^2/\lvert G_U\rvert^2$	0.653	0.142	8.28×10^{-3}	1.68×10^{-5}
$e^{-2\pi\beta}$	1.000	0.533	0.343	0.137

TABLE 7.5 The estimated reflection coefficient for gravity waves when $k_x = 0.55$ and $\theta = 30^o$. It is calculated as a function of the asymptotic vertical wave number of the gravity mode, $\beta^2 = k_x^{\,2}(\frac{N^2}{\sigma^2} - 1) - \frac{1}{4}$ The upgoing Alfvén- wave has been eliminated by linearly combining the solutions S_1 and S_2 and shows that, in the limit of long vertical wave length, gravity energy is partially reflected in the same mode. The final line gives comparative values for reflection in a viscous medium.

Horizontal Magnetic Field

For completeness, one further example needs to be considered. That is when the magnetic field is horizontal. This is a special case of that studied by Rudraiah and Venkatachalappa (1972) for rotating fluids. Here the differential equation reduces to the form

$$(1 - \sigma_a^{\,2}/\sigma^2)\frac{d_2w}{dz^2} - \frac{dw}{dz} + k_h^{\,2}\left[\frac{N^2}{\sigma^2} - 1 + \frac{\sigma_a^{\,2}}{\sigma^2}\right]w = 0$$

$$\sigma_a^{\,2} = c_a^{\,2}k_x^{\,2} \qquad (7.21)$$

The interpretation of this equation in terms of mode conversion may be readily seen from the dispersion equation which in the Boussinesq limit has the form

$$k^2(\sigma^2-\sigma_a^{\ 2}) \quad = \quad k_h^{\ 2}N^2 \tag{7.22}$$

whence

$$\lim_{\sigma_a^{\ 2}\to\sigma^2} k^2 \quad = \quad \infty \tag{7.23}$$

The expression for the group velocity given by equations (4.13 to (4.15) reduces to

$$V_{gx} \quad = \quad c_a^{\ 2}k_x/\sigma \qquad V_{gy} \quad = \quad V_{gz} \quad = \quad 0; \qquad \sigma^2 = \sigma_a^{\ 2} \tag{7.24}$$

An upgoing gravity wave is therefore converted to a horizontally travelling Alfven wave at the level defined by $c_a^{\ 2}k_x^{\ 2} = \sigma^2$ and is trapped at this level. Infinite wave amplitudes result.

Figure (7.4) gives the numerical integration of equation (7.21) when $k_x = 4.25$ and $N/\sigma = 1.2$. Since the differential equation has a singularity at the trapping height it is not amenable to numerical integration in the vicinity of the singularity. Following Jones (1967) we add Rayleigh viscosity to bypass the singularity. This is included by a transformation on σ,

$$\sigma^2 \quad \to \quad \sigma^2 - i\varepsilon. \tag{7.25}$$

By this method the singularity is removed by the addition of a small viscous drag; in this case ε/σ^2 is set at 10^{-2}.

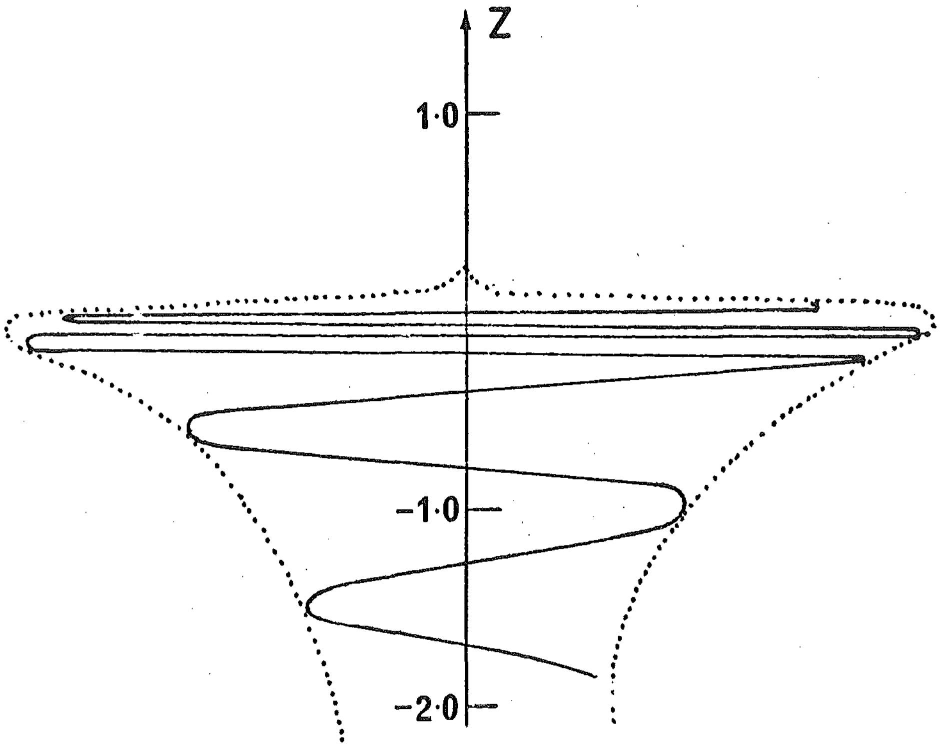

FIGURE 7.3

The wave function calculated for a horizontal ambient magnetic field and plotted against reduced height. Other wave parameters are, $k_x = 4.25$, $N/\sigma = 1.2$. Critical behaviour is present at $z = 0$ and integration over this level has been achieved by the inclusion of Rayleigh damping. The dotted envelope shows the wave amplitude.

The behaviour of the dispersion equation near $\theta = 0$ has been discussed in Chapter 4, see equation (4.19). The solution calculated here is a continuation of S_2 by virtue of the continuity of the dispersion equations. The expansion of S_1 has the leading term exp $(-k_x \cot \theta)$ which becomes meaningless near $\theta = 0$.

The wave solutions indicate wave properties at variance with those deduced from ray tracing methods (Chapter 5). Wave solutions show no change in behaviour near the critical angle implied in equation (4.22). All gravity wave energy has been converted to energy in the magnetic mode irrespective of the upgoing wave's orientation to the ambient magnetic field. We are forced to the conclusion that ray tracing methods have in this case given misleading results. The Boussinesq approximation used as the basis of ray tracing replaces the operator $\partial_2/\partial S^2$ by $-k_s{}^2$. This is a bad approximation. Secondly, under those conditions where the wave solutions and ray tracing are in disagreement, the plots of k_z versus z approach a cusp at zero density. Moreover in this region, modes at the same frequency and with comparable wave numbers travel in opposite directions. A process analogous to coupling can be expected thus preventing the flow of energy through the cusp.

P A R T I I

SOME TENTATIVE EXTENSIONS

Here the less important properties of hydromagnetic waves are presented. Some extensions are made to media having different properties and the possible application of these extensions explored. Material in this Section may be regarded as more speculative.

CHAPTER 8

THE BOUSSINESQ APPROXIMATION AND WAVE SOLUTIONS

We now compare wave solutions with the properties deduced from the dispersion equation in the Boussinesq limit. Figure (7.1) plots the level of mode conversion for a wave having a horizontal component of wave vector with a constant amplitude. Here $k_h = 0.55$. The ratio of the Brunt frequency to the wave frequency is, $N/\sigma = 5.0$. The parameter ϕ measures the angle between the magnetic meridian and the horizontal component of the wave vector while θ is the angle of inclination of the magnetic field vector. For convenience the definition of the angle θ will be extended. Both the differential equation and its associated dispersion equation is of even order on k_h while the angle θ enters only through the expansion of the operator $\partial_2/\partial s^2$ (equation 4.3). A negative value of k_x is equivalent to changing the sign of $\cos\theta$. Functions therefore become single-valued if a negative value of k_x is interpreted as a wave propagating at the supplementary angle of θ. The contours plotted are isopleths of the level of mode conversion expressed in units of the atmospheric scale height according to the relationship $z = \log_e(c_a{}^2 k_h{}^2/\sigma^2)$. Relativity of the height scale is preserved by this choice of height parameter. The plot shows the importance of the parameter $(N^2/\sigma^2-1)k_h{}^2-k_x{}^2\cot^2\theta$; see equation (4.11). The isopleth defined by a zero value of this parameter is represented by the boundary of the hatched area in Figure (8.1), and gives the conditions under which a ray will just penetrate to

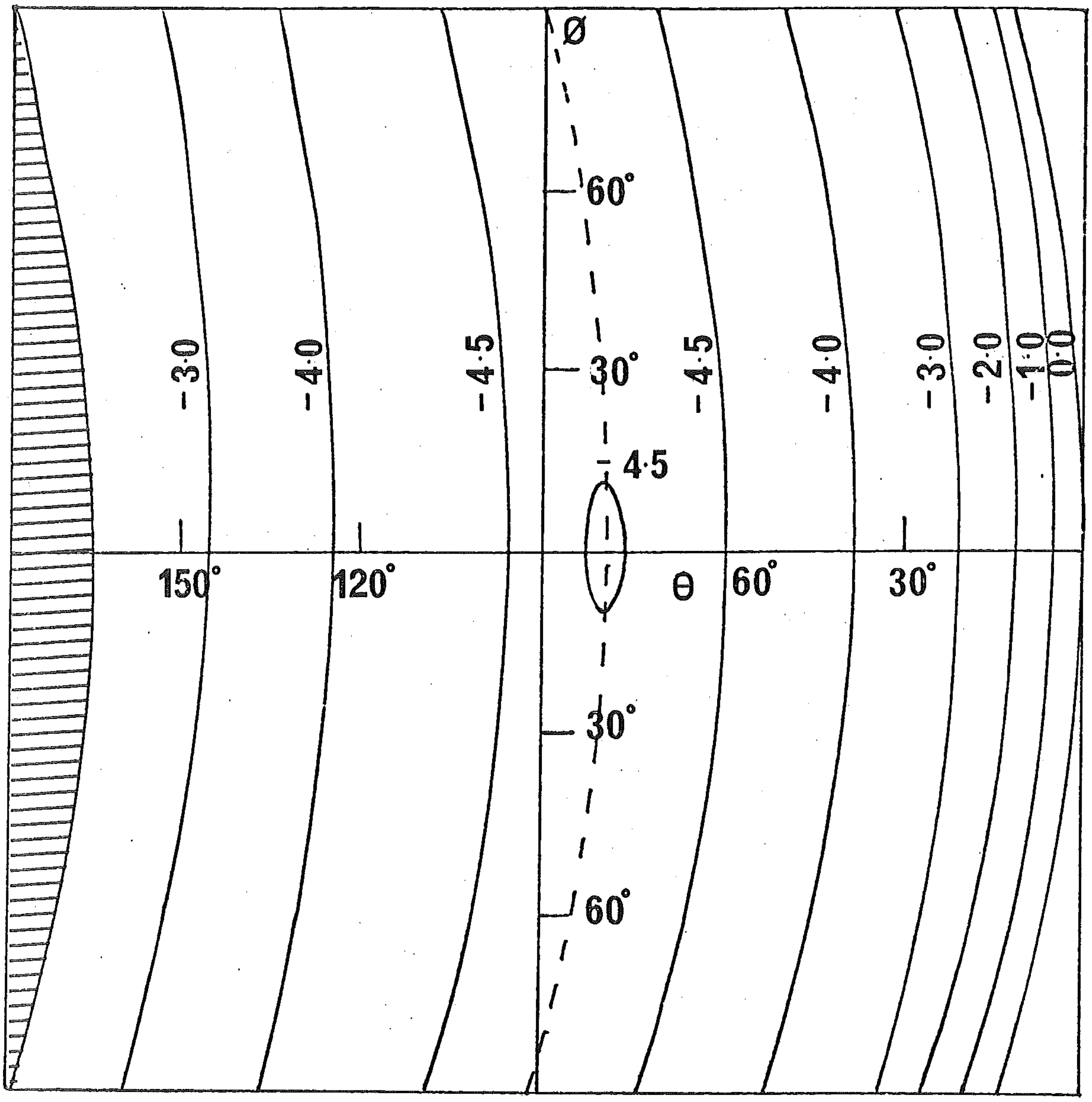

FIGURE 8.1

The level of mode conversion derived from the dispersion equation in the Boussinesq limit. Wave parameters are $|k_h| = 0.55$, $N/\sigma = 5.0$. The angle θ measures the inclination of the magnetic field vector while ϕ measures the angle between $\underline{k}_h$ and the magnetic meridian.

infinity in the Boussinesq approximation. The critical angle defined by $\phi = 0$, $\cot^2\theta_c = N^2/\sigma^2 - 1$ is $\theta = 167.7^\circ$ and a gravity wave with corresponding initial wave parameters will penetrate to infinity, motion will at all times be along the direction of the magnetic field line and unaffected by its presence. Alternatively if the initial motion is in the magnetic meridian and perpendicular to the magnetic field line then the effects are greatest and the level of mode conversion is a minimum. The point $\theta = 77{,}4^\circ$ is thus enclosed by contours. The plot near $\theta = 90^\circ$ has more structure than that shown in Figure (8.1). The line $\theta = 90^\circ$ is an isopleth since the level of mode conversion is here independent of ϕ. Contours just to the left of $\theta = 90^\circ$ lose their curvature while those to the right are closed for $90^\circ < \theta < \theta_c + 90^\circ$. The dotted contour represents the minimum level of conversion for ϕ fixed and is a repetition of the curve satisfying $(N^2/\sigma^2 - 1)k_h{}^2 - k_x{}^2\cot^2\theta = 0$ displaced by 90°. The hatched area represents conditions in which the WKB approximation is invalid.

The wave solutions show no special signature which may be assigned to a specific level to signify the level of mode conversion. However, some indication of this level may be obtained from the plots of the amplitude versus height of the decomposed wave. Figure (8.2) plots these functions. Here $\log_e|w|$ of the wave solution broken into its component parts is plotted against the reduced height defined by $z = \log_e(c_a{}^2 k_h{}^2/\sigma^2)$. Relativity of amplitudes between wave pairs has been preserved even though the amplitude scale is expressed in arbitrary units. Plots are given for solutions with parameters $k_x = 0.55$, $k_y = 0.0$, $N/\sigma = 5.0$ for each

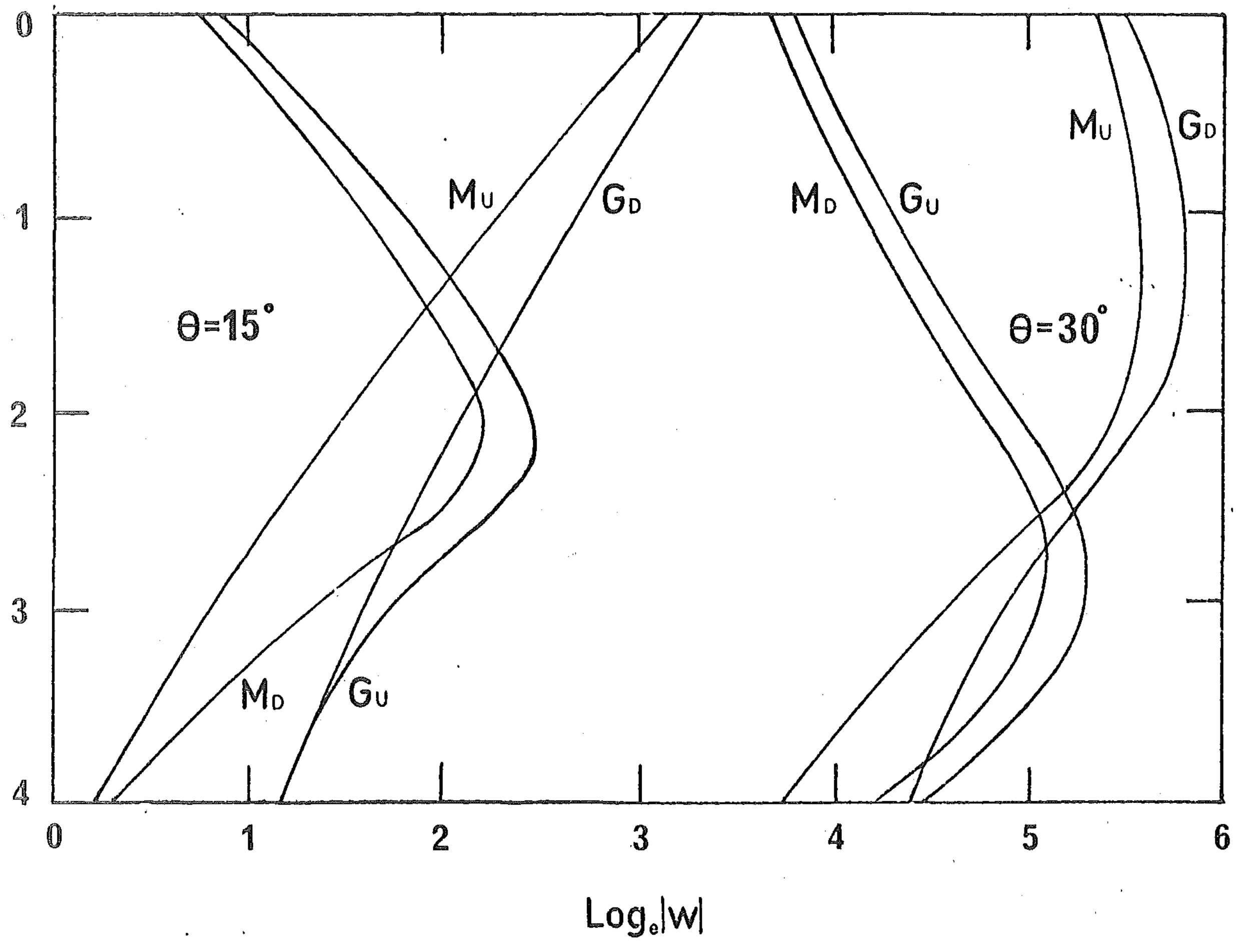

FIGURE 8.2

The variation of amplitude with reduced height derived from the decomposition of wave solutions into component parts. Wave parameters are $k_x = 0.55$, $k_y = 0.0$, $N/\sigma = 5.0$. Plots are for two inclinations of the magnetic field, $\theta = 15^{\circ}$ and $\theta = 30^{\circ}$. The scale of $\log_e|w|$ is arbitrary; the plots have been separated to avoid confusion.

inclination of magnetic field $\theta = 15^{\circ}$ and $\theta = 30^{\circ}$. The symbols G and M signify the magnetic and gravity modes while the suffix indicates the direction of energy propagation of the equivalent wave system. The amplitude of wave pairs have maxima at the same level even though of different amplitudes. This maximum is taken to indicate the level of mode conversion.

Figure (8.3) compares the height of maximum amplitude with the level of mode conversion indicated by the dispersion equation in the Boussinesq limit. This comparison is made for a cross-section of Figure (8.1) and is taken along the axis $\theta = 0^{\circ}$. It will be seen that the dispersion equation gives a reasonable indication of the behaviour of the wave solutions even though the level of mode conversion may be underestimated by as much as one scale height.

So far we have used the zero order WKB approximation to interpret the wave solutions and have done so without reference to the accuracy of this approximation. Equation (4.2) is a fourth order differential equation which does not readily admit higher order WKB approximations. We therefore content ourselves to examining the limits to which the zero order approximation holds. Writing,

$$w = e^{i\int k_z dz} \tag{8.1}$$

then

$$w'' = -(k_z^2 - ik_z')w \tag{8.2}$$

Higher derivatives become complicated. A necessary condition for validity is,

$$k_z^2 \gg ik_z' \tag{8.3}$$

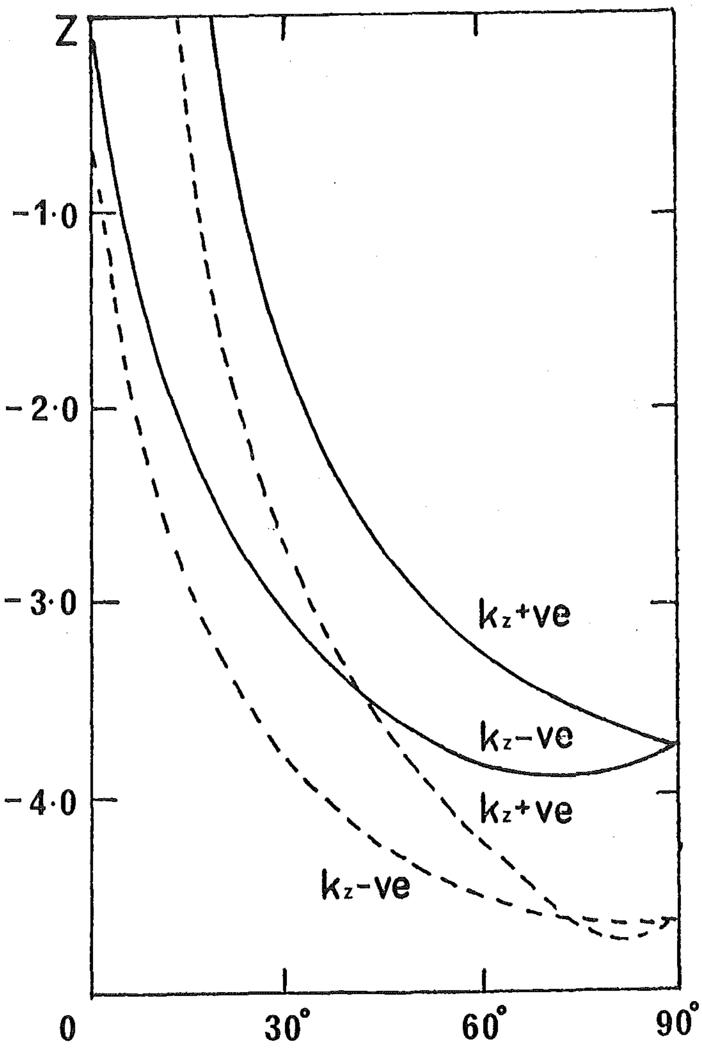

FIGURE 8.3

This plots the variation of the height of maximum amplitude for waves where $N/\sigma = 5.0$, $k_x = 0.55$, $k_y = 0.0$. The gravity and magnetic modes having k_z of the same sign show coincident maxima. This indicates the level of mode conversion. The hashed curves indicate the level of mode conversion deduced from the dispersion equation in the Boussinesq limit.

A further restriction exists. Equation (4.2) contains the differential operator $\partial_2/\partial s^2$ while one solution near infinity is $w \approx \exp(ik_x \cot\theta\, z)$. Under these conditions $\partial_2/\partial s^2$ is of order zero and the WKB approximation becomes invalid. The interpretation in terms of elemental waves hence has no meaning. In Figure (7.2) the wave decomposition is taken only to $z = 0$; $|k_s'/k_s^2| < 0.25$ throughout the plotted curves. Table (8.1) itemizes values of this parameter for the two plots given in Figure (8.3).

z	0.0	-1.0	-2.0	-3.0	-4.0
$\theta = 15^o$	.081	.020	.017	.006	.003
$\theta = 30^o$	.216	.141	.067	.030	.018

TABLE 8.1. Tabulates $|k_z'/k_z^2|$ as a function of z for the downgoing gravity wave. This represents the maximum error encountered when applying the zero order WKB approximation.

CHAPTER 9

WAVES IN A COMPRESSIBLE MEDIUM

(HORIZONTAL MAGNETIC FIELD WITH MERIDIONAL MOTION)

So far we have considered waves in an incompressible medium. We must now ask the question 'Does the established wave behaviour also apply to a compressible medium?' Now, in a pure hydrodynamic medium the differential equation describing the perturbations in an incompressible medium may be shown equivalent to its compressible counterpart as the limit when the velocity of sound becomes infinite (Yih, 1965, p.16). This limit is well attained by waves whose horizontal trace velocity is small compared with the velocity of sound. A hydromagnetic medium however is characterized by two velocities, the velocity of sound and the Alfvén velocity. An infinite sonic velocity presupposes that the velocity of sound exceeds the Alfvén velocity, a condition only encountered in highly ionized atmospheres where the temperature is high. It is not clear that the behaviour so far established for $c_a < c_0$ has comparable behaviour when $c_a > c_0$. In the latter case, the equations of motion must of necessity include both these velocities; a big increase in mathematical complexity will result. See McLellan and Winterberg (1968). Here we endeavour to avoid complexities and confine our attention to the propagation of waves in the plane of the magnetic meridian when the magnetic field is horizontal.

The equation expressing the continuity of the magnetic field is (Chandrasekhar, 1961),

$$\frac{\partial \underline{h}}{\partial t} = (\underline{H}\cdot\nabla)\underline{u} - \psi\underline{H} \tag{9.1}$$

where the fluid divergence no longer vanishes

$$\psi = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \neq 0 \tag{9.2}$$

The equation of motion of the fluid is as before,

$$\frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p + \frac{\mu}{\rho_0}(\nabla\times\underline{H})\times\underline{H} - \frac{g\rho}{\rho_0} \tag{9.3}$$

Eliminating $\underline{h}$ by differentiating equation (9.3) with respect to time and then substituting equation (9.1) we derive,

$$\frac{\partial^2 \underline{u}}{\partial t^2} = -\frac{1}{\rho_0}\nabla\frac{\partial p}{\partial t} + \frac{\mu}{\rho_0}\left[\nabla\times\{(\underline{H}\cdot\nabla)\underline{u}-\psi\underline{H}\}\right]\times\underline{H} - \frac{g}{\rho_0}\frac{\partial \rho}{\partial t}\hat{k} \tag{9.4}$$

Expansion of the magnetic terms gives

$$\frac{\mu}{\rho_0}\{\nabla\times(\underline{H}\cdot\nabla)\underline{u}\}\times\underline{H}$$

$$= c_a^{\,2}\frac{\partial}{\partial s}\left[\sin\theta\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right);\cos\theta\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-\sin\theta\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right);\right.$$

$$\left.-\cos\theta\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\right] \tag{9.5}$$

and

$$\frac{\mu}{\rho_0}\{\nabla\times(\psi\underline{H})\}\times\underline{H}$$

$$= c_a^{\,2}\left[\sin\theta\cos\theta\frac{\partial\psi}{\partial z} - \sin^2\theta\frac{\partial\psi}{\partial x};\ -\frac{\partial\psi}{\partial y};\ -\cos^2\theta\frac{\partial\psi}{\partial z}\right.$$

$$\left.+\sin\theta\cos\theta\frac{\partial\psi}{\partial x}\right] \tag{9.6}$$

The perturbation equations for the behaviour of the fluid now become

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{9.7}$$

$$\frac{\partial^2 w}{\partial t^2} - c_a^2\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right] = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial t} - \frac{g}{\rho_0}\frac{\partial \rho}{\partial t} \tag{9.8}$$

$$\frac{\partial p}{\partial t} + w\frac{dp_0}{dz} - c_0^2\left(\frac{\partial \rho}{\partial t} + w\frac{d\rho_0}{dz}\right) = 0 \tag{9.9}$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\rho_0}{dz} + \rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0 \tag{9.10}$$

While the constitutive relations for a compressible gas are,

$$\frac{dp_0}{dz} = \frac{c_0^2}{\gamma}\frac{d\rho_0}{dz} = -\rho_0 g \tag{9.11}$$

$$\frac{d\rho_0}{dz} = -\frac{\gamma g \rho_0}{c_0^2} = -\frac{\rho_0^2}{p_0}g \tag{9.12}$$

With the aid of equation (9.7) and (9.10) we eliminate ρ and u from equation (9.9) to give

$$\frac{\partial^2 p}{\partial t^2} - c_0^2\frac{\partial^2 p}{\partial x^2} = \frac{\rho_0^2}{p_0}g\frac{\partial w}{\partial t} - \rho_0 c_0^2\frac{\partial^2 w}{\partial z \partial t} \tag{9.13}$$

Thus in the limit where the vertical component of velocity vanishes, equation (9.13) represents a sound wave travelling along the horizontally directed magnetic field line. The velocity of the wave is that of sound and the ambient magnetic field enters into the expression only in so far as it modifies the ambient pressure.

Using equation (9.7) and (9.9), equation (9.8) becomes,

$$\rho_0 \frac{\partial}{\partial t}\left[\frac{\partial^2 w}{\partial t^2} - c_a{}^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g\frac{\partial w}{\partial z} + \frac{\gamma g^2}{c_0{}^2} w\right]$$

$$= -\frac{\partial}{\partial z}\left(\frac{\partial^2 p}{\partial t^2}\right) + g\frac{\partial^2 p}{\partial x^2} \tag{9.14}$$

while equation (9.13) may be rewritten as,

$$\rho_0 \frac{\partial}{\partial t}\left[\frac{\rho_0}{p_0} gw - c_0{}^2 \frac{\partial w}{\partial z}\right] = \frac{\partial^2 p}{\partial t^2} - c_0{}^2 \frac{\partial^2 p}{\partial x^2} \tag{9.15}$$

These equations may be written in a more convenient form,

$$\rho_0 A = \frac{\partial P}{\partial z} - gQ \tag{9.16}$$

$$\rho_0 B = P - c_0{}^2 Q \tag{9.17}$$

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 Q}{\partial t^2} \tag{9.18}$$

where

$$A = -\frac{\partial}{\partial t}\left[\frac{\partial^2 w}{\partial t^2} - c_a{}^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - g\frac{\partial w}{\partial z} + \frac{\gamma g^2}{c_0{}^2} w\right] \tag{9.19}$$

$$B = \frac{\partial}{\partial t}\left(\frac{\rho_0 g}{p_0} w - c_0{}^2 \frac{\partial w}{\partial z}\right) \tag{9.20}$$

$$P = \frac{\partial^2 p}{\partial t^2} \quad ; \qquad Q = \frac{\partial^2 p}{\partial x^2} \tag{9.21}$$

The required differential equation is thus,

$$\frac{\partial^2}{\partial t^2}\left[A - \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 B)\right] = \frac{\partial^2}{\partial x^2}\left[c_0{}^2 A - gB\right] \tag{9.22}$$

It will be noted that this equation is linear in A and B and hence may be integrated with respect to time. Let this step be implied. It will be further noted that equation (9.22) admits the Boussinesq approximation,

$$\frac{\partial^2}{\partial t^2}\left[A - \frac{\partial B}{\partial z}\right] = \frac{\partial^2}{\partial x^2}\left[c_0^{\,2}A - gB\right] \tag{9.23}$$

The final differential equation is then,

$$\sigma^2\left(1 + \frac{c_a^{\,2}}{c_0^{\,2}} - \frac{\sigma_a^{\,2}}{\sigma^2}\right)\frac{d^2w}{dz^2} - \sigma^2\,\frac{\gamma g}{c_0^{\,2}}\,\frac{dw}{dz} + k_x^{\,2}\left[(N^2-\sigma^2+\sigma_a^{\,2})\left(1 - \frac{\sigma^2}{c_0^{\,2}k_x^{\,2}}\right) + \frac{N^2\sigma^2}{c_0^{\,2}k_x^{\,2}}\right]w = 0 \tag{9.24}$$

where N^2 is the 'compressible' Brunt frequency

$$N^2 = \frac{(\gamma-1)g^2}{c_0^{\,2}}$$

In the limit where the Alfven velocity and its associated frequency become zero, equation (9.24) approaches the well known differential equation of a gravity wave in a stratified non-conducting atmosphere,

$$\sigma^2\,\frac{d^2w}{dz^2} - \sigma^2\,\frac{\gamma g}{c_0^{\,2}}\,\frac{dw}{dz} + \left[\frac{\sigma^4}{c_0^{\,2}} + k_x^{\,2}(N^2-\sigma^2)\right]w = 0 \tag{9.25}$$

It is not the purpose of this chapter to analyse equation (9.24) but merely to point out that the behaviour of an incompressible fluid represents a realizable asymptote of the likely behaviour of a compressible fluid. For gravity waves $c_0^{\,2}k_x^{\,2}/\sigma^2 > 1$ and in terms of this variable equation (9.24) becomes,

$$\sigma^2\left[1 - \frac{c_a^2}{c_0^2}\left(\frac{c_0^2 k_x^2}{\sigma^2} - 1\right)\right]\frac{d_2 w}{dz^2} - \sigma^2 \frac{\gamma g}{c_0^2}\frac{dw}{dz}$$

$$+ k_x^2\left[N^2 + (\sigma_a^2 - \sigma^2)\left(1 - \frac{\sigma^2}{c_0^2 k_x^2}\right)\right]w = 0 \qquad (9.26)$$

Clearly, in the limit of an infinite velocity of sound equation (9.26) assumes the form of its incompressible asymptote given by equation (7.21) while only when $c_0^2 k_x^2/\sigma^2 > 1$ may the coefficient of d_2w/dz^2 become zero and result in critical wave behaviour.

One important conclusion can be drawn from the dispersion relation associated with equation (9.26). In the Boussinesq limit,

$$\sigma^2\left[1 - \frac{c_a^2}{c_0^2}\left(\frac{c_0^2 k_x^2}{\sigma^2} - 1\right)\right]k_z^2 - \left[N^2 + (\sigma_a^2 - \sigma^2)\left(1 - \frac{\sigma^2}{c_0^2 k_x^2}\right)\right]k_x^2$$

$$= 0 \qquad (9.27)$$

At the critical level where the coefficient of k_z^2 becomes zero, then k_z^2 becomes infinite in order that the relation be satisfied. The components of the group velocity here become,

$$v_{gz} = 0$$

$$v_{gx} = \left(\frac{1}{c_0^2} + \frac{1}{c_a^2}\right)^{-\frac{1}{2}} \qquad (9.28)$$

and the wave velocity tends to that of a sound wave or an Alfvén wave, whichever is the smaller.

CHAPTER 10

PROPERTIES OF HYDROMAGNETIC WAVES

In this chapter we briefly summarize the properties of waves in a perfect hydromagnetic medium. In an incompressible medium, the equation of motion may be separated into a wave equation representing the slow Alfvén mode and a fourth order wave equation which represents hybrid waves having the character of both Alfvén waves and gravity waves. These waves were called 'inertio-magnetic' waves by Rudraiah and Vankatachalappa (1972) or they may be called 'gravitio-magnetic' waves.

The slow Alfvén mode in a stratified atmosphere has the properties deduced for the classical slow Alfvén mode; motion is at all times horizontal and gravitational forces do not operate. The wave propagates by means of a partition of energy between the kinetic energy of motion and the energy stored by deforming the magnetic field. Its velocity is independent of frequency and depends only on the ambient magnetic pressure of the medium and its density.

Propagating inertio-magnetic waves share their internal energy between at least three internal forms. The kinetic energy of motion, the energy gained by disturbing the equilibrium of a stratified fluid in a gravitational field and the energy introduced by deforming the magnetic field. Clearly with these alternative forms of energy, the possible wave motions of the medium must have complex properties, particularly when the ambient properties of the medium perforce vary . No simple partition of energy is possible. A non-separable wave equation mirrors this greater complexity.

If the fluid is compressible, then the internal energy due to the compression of the fluid must also be included and a further degree of complexity is added. Here any factorization into more elemental forms of wave equation seems unlikely. The four asymptotic approximations, for fast and slow Alfvén waves, gravity waves and sound waves, certainly exist in this type of atmosphere but the inter-relation and the possible forms of mode conversion between waves presents considerable analytic difficulties.

Expressions for the partition of the internal energy can be found from equation (3.21). The terms contained within the first pair of brackets when averaged over one wave cycle give the magnitude of these quantities.

$$\langle E\rangle = \tfrac{1}{2}\rho_0\langle \underline{u}\cdot\underline{u}\rangle + \tfrac{1}{2}\mu\langle \underline{h}\cdot\underline{h}\rangle + \tfrac{1}{2}\rho_0 N^2\langle \zeta^2\rangle$$

$$\begin{bmatrix}\text{TOTAL}\\ \text{ENERGY}\end{bmatrix} = \begin{bmatrix}\text{KINETIC}\\ \text{ENERGY}\end{bmatrix} + \begin{bmatrix}\text{MAGNETIC}\\ \text{ENERGY}\end{bmatrix} + \begin{bmatrix}\text{POTENTIAL}\\ \text{ENERGY}\end{bmatrix} \tag{10.1}$$

Consider first the gravity wave asymptote. When ρ_0 is large, the term $\tfrac{1}{2}\mu\langle \underline{h}\cdot\underline{h}\rangle$ can be neglected from the expression and a simple partition exists between the kinetic and potential energies of the wave. Furthermore, in this asymptote, the energy is uniformly distributed in space. Similarly in the Alfvén asymptote, motion becomes horizontal and the potential energy term becomes negligible. There is here a partition between the kinetic energy and the energy stored in the deformed magnetic field. This is as it should be, and such is easily verified by using the expression for $\underline{h}$ given by equation (3.20) to calculate this.

$$\begin{bmatrix}\text{MAGNETIC}\\ \text{ENERGY}\end{bmatrix} = \tfrac{1}{2}\rho_0 c_a^{\,2} \left\langle \frac{\partial \underline{\xi}}{\partial s} \cdot \frac{\partial \underline{\xi}}{\partial s} \right\rangle$$

$$= \tfrac{1}{2}\rho_0 \frac{c_a^{\,2} k_s^{\,2}}{\sigma^2} \langle \underline{u} \cdot \underline{u} \rangle \qquad (10.2)$$

The second form of the expression is written to the order of accuracy of the WKB approximation. Now in the Alfvén limit $c_a^{\,2} k_s^{\,2}/\sigma^2 = 1$ and partition between the kinetic and magnetic densities is therefore maintained. Motion in this asymptote tends towards the horizontal direction and the total kinetic energy density therefore depends only on the horizontal component of velocity. Since $u \propto \frac{\partial w}{\partial z}$ then the kinetic energy density varies as $\rho_0^{\,\frac{1}{2}}$ whereas the potential energy density depends upon w and is distributed as $\rho_0^{\,-\frac{1}{2}}$. The ratio of magnetic to potential energy density therefore varies directly as the density ρ_0.

As already noted, the energy distribution for gravitio-magnetic waves is not uniform and no simple partition exists between the two forms of potential energy. For an upgoing gravity wave, the kinetic energy density approximately varies as $\tfrac{1}{2}\rho_0 w^2 k^2/k_h^{\,2}$ while the gravitational potential energy is approximately $\tfrac{1}{2}\rho_0 w^2 N^2/\sigma^2$. Both of these quantities increase with height. Since k^2 also increases with height then the kinetic energy density increases more rapidly than the potential form. The magnetic energy correspondingly increases.

After conversion to an Alfvén wave both kinetic and magnetic densities increase while the potential energy diminishes.

The principal feature of the behaviour of gravitio-magnetic waves is the conversion of the gravity mode into the Alfvén mode. Since the differential equation is not amenable to analytic solution numerical methods of solution have been used and mode conversion demonstrated by the numerical properties of the solution. WKB wave approximations are matched to the wave solutions well below the level where the mode conversion occurs. The behaviour pattern has therefore been established to only the order of accuracy of the WKB approximation. This, together with the numerical difficulties encountered when the vertical wave number is small, precludes satisfactory checking of the hypothesis of complete mode conversion for frequencies near the Brunt frequency. That the differential equation does not appear to factorize into separate wave equations each containing directionally alternate Alfvén and gravity modes suggests that complete mode conversion is merely a good approximation valid in those regions where the WKB approximation is valid. Indeed solutions for $N^2/\sigma^2 \approx 1$ suggest that reflection in the same mode is possible. This result might be expected on physical grounds. For a gravity wave with a frequency near the Brunt frequency both the group velocity and the transmitted energy flux tend to zero. These quantities do not vanish at these frequencies in the Alfvén limit. Complete mode conversion is therefore impossible at frequencies near the Brunt frequency.

So far we have sought to clarify the properties of waves by assuming that their amplitude depends only on the single height parameter z. Consequently solution is limited to the asymptotic limit of plane harmonic waves. We have thus established behaviour only in the steady state and have completely overlooked the possibility that this steady state does not of necessity represent a physically realizable asymptote. The stability of these waves requires that the steady state may be attained and only when this condition is satisfied may waves be considered to have the properties here described.

Time dependent solutions of the propagation of gravity waves initiated by impulsive displacements have been found. Tolstoy (1973, p289) considers the propagation from a point source in an incompressible gas. The asymptotic behaviour is that deduced from the solution in the steady state. Booker and Bretherton (1967) discussed the behaviour of incompressible waves in an atmosphere containing a background wind and reached a similar conclusion. Both these studies dealt with second order wave equations in which analytical solutions were known. However, for gravitio-magnetic waves, the wave equation is of the fourth order without known solutions in closed form. The methods used by Tolstoy, Booker and Bretherton do not apply in this case and the question of stability therefore is left unanswered. On the other hand, the numerical method of analysis given by Houghton and Jones (1969) may well find ready application when deciding questions of stability.

CHAPTER 11

RAY TRACING IN DISSIPATIVE MEDIA

Weinberg (1962) discusses the eikonal method of ray tracing in magnetohydrodynamic media. When discussing the exclusion of dissipative effects he states; 'It should be emphasized that this is not an essential omission; such effects could easily be handled'. It is of interest to apply these methods to gravity wave equations with known solutions and examine the implicit restrictions of the method. Here we shall assume no prior knowledge of the field variables associated with the differential equations describing the perturbations. We note that Weinberg's treatment presupposes the Boussinesq approximation since the zero order WKB approximation is made in the primitive equations. He has neglected gravitational effects and his equations are already expressed in terms of field variables; the partition of energy between mechanical and magnetic is directly proportional to the square of the perturbation velocity and the square of the magnetic perturbations. The density of the medium does not enter into the expressions. The necessity of this restriction is examined by example.

Consider a gravity wave in an incompressible medium. The differential equation describing the vertical component of the perturbation velocity is, in its primitive form,

$$\frac{1}{\rho_0}\frac{d}{dz}\left(\rho_0\frac{dw}{dz}\right) + k_h{}^2\left(\frac{N^2}{\sigma^2} - 1\right)w = 0 \qquad (11.1)$$

which leads directly to the dispersion relation,

$$k_z^{\,2} + i\beta k_z = k_h^{\,2}\left(\frac{N^2}{\sigma^2} - 1\right) \tag{11.2}$$

$$\beta = -\frac{1}{\rho_0}\frac{d\rho_0}{dz} .$$

Direct application of the eikonal equations to this dispersion relationship gives,

$$\frac{dk_x}{dt} = 0, \quad \frac{dk_y}{dt} = 0, \quad \frac{dk_z}{dt} = 0$$

$$\begin{aligned} \frac{dz}{dt} &= -\frac{\sigma(k_z+\frac{1}{2}i\beta)}{k_h^{\,2}+k_z^{\,2}+i\beta k_z} \\ &= -\frac{\sigma(k_z+\frac{1}{2}i\beta)}{k_h^{\,2}N^2/\sigma^2} \\ &= -\frac{\sigma \mathrm{Re}(k_z)}{k_h^{\,2}\,N^2/\sigma^2} \end{aligned} \tag{11.3}$$

The final expression for the vertical component of the group velocity corresponds to that derived by the usual method. It will be noted that the field variables are here of a simple form and they could be established by this method. We also note that this expression could have been derived either by neglecting the imaginary term containing k_z or by rewriting the dispersion equation as,

$$(k_z + \tfrac{1}{2}i\beta)^2 = k_h^{\,2}\left(\frac{N^2}{\sigma^2} - 1\right) - \frac{\beta^2}{4} \tag{11.4}$$

We now confine our attention to gravity waves in a viscous medium. Following Yanowitch (1967), the differential equation describing the perturbation of the stream function ψ is,

$$|\rho_0\psi_{xx} + (\rho_0\psi_z)_z|_{tt} - g\rho_0{}'\psi_{xx} = \mu\Delta\Delta\psi_t, \tag{11.5}$$

The dispersion relationship in the Boussinesq approximation is then,

$$k^2\sigma^2 - N^2k_x{}^2 + \frac{i\sigma\mu}{\rho_0} k^4 = 0. \tag{11.6}$$

Here k is dimensionless and the coefficient of viscosity μ is referred to H^2. Rewriting this as,

$$k^2\left[(\sigma + \tfrac{1}{2}\frac{i\mu}{\rho_0} k^2)^2 + \tfrac{1}{4}\frac{\mu^2k^4}{\rho_0{}^2}\right] - N^2k_x{}^2 = 0, \tag{11.7}$$

then expressions for the rate of change of wave number and the position of a waves packet are

$$\frac{dk_z}{dt} = \frac{1}{4\sigma} k^4 \frac{\mu^2}{\rho_0{}^2}\left(-\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right) \tag{11.8}$$

$$\frac{dz}{dt} = -\frac{k_z}{\sigma k^2}(\sigma^2 + \frac{4}{4}\frac{\mu^2}{\rho_0{}^2} k^4). \tag{11.9}$$

Here we have disposed of the imaginary term in the dispersion equation by treating the frequency σ as a complex quantity. This is only justified when viscous effects are small. However, these effects depend on the kinematic viscosity μ/ρ_0 and since this quantity increases exponentially with height a rapid onset of viscous effects is to be expected. We now find wave behaviour when μ/ρ_0 is not small, we extrapolate into regions where viscosity dominates.

To interpret these equations we note that the dispersion equation restricts k; since μ/ρ_0 increases with height then k_z decreases. Further $k_z = 0$ is at a finite height defined by a density according to the expression,

$$k_x^{\,4} = \tfrac{1}{4}\frac{\rho_0^{\,2}}{\mu^2}(N^2-\sigma^2). \tag{11.10}$$

For convenience this height may be defined as an effective 'reflection level' for the wave. Consistency of equations (11.8) and (11.9) requires a negative value of k_z corresponding to an up-flow of energy. $|k_z|$ then decreases as the packet moves higher in the atmosphere until the group velocity becomes zero. The subsequent history of the packet can be traced by noting that $(dk_z/dt)_{z=z_r}$ is non-zero at the reflection level and continuous behaviour can be expected. We use equation (11.7), with only the real part of σ retained, to eliminate ρ_0 from equation (10.8). Integrating this result then,

$$\rho\beta t = (k_z'-k_z) + \left(\frac{1}{2\ell}\right)^{k_x}\frac{N^2}{\sigma^2}\log\left[\frac{\ell k_x+k_z}{\ell k_x+k_z'}\cdot\frac{\ell k_x-k_z'}{\ell k_x-k_z}\right], \tag{11.11}$$

where $\ell = \left(\dfrac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}}$

$k_z' = k_z$ at $t = 0$.

Setting $k_z = 0$ we note that a packet reaches the reflection level in a finite time. The ray trace has therefore indicated a reflection process.

Yanowitch has computed wave solutions to equation (11.5) for a range of parameters. He has not assumed a Boussinesq limit and has defined the following variables.

$$r = k_h^{\,2}\left(\frac{N^2}{\sigma^2} - 1\right) \qquad (11.12)$$

$$s = \tfrac{1}{2}\log \tfrac{1}{4}\,\frac{\mu}{\rho_0\sigma} \qquad (11.13)$$

Clearly s is a convenient height parameter. Figure (11.1) reproduces Yanowitch's plots of $s|U|$ versus s. Since U is the horizontal component of the perturbation velocity, then $s|U|$ will be constant in the region where propagation is uni-directional and unaffected by viscosity. A reflected wave will modulate this variable, the amplitude of modulation depending on the reflection coefficient. A decrease in the amplitude of this variable will indicate the level where this reflection occurs. Figure (11.1) shows that reflection takes place within a scale height of $s = 0$.

Using equation (10.12) and (10.13) then the reflection condition given by equation (10.10) becomes,

$$s = \tfrac{1}{2}\log\left(\frac{1}{8k_x^{\,3}}\, r^{\frac{1}{2}}\right) \qquad (11.14)$$

The circles plotted in Figure (11.1) give the reflection levels indicated by this relation. Clearly, the Boussinesq approximation has given greater wave penetration than that indicated by the wave solution. If we judge that reflection has taken place at the half power point of $s|U|$ then the reflection level is $s = 0.61$ for $r = 0.5$ and $s = 0.88$ for

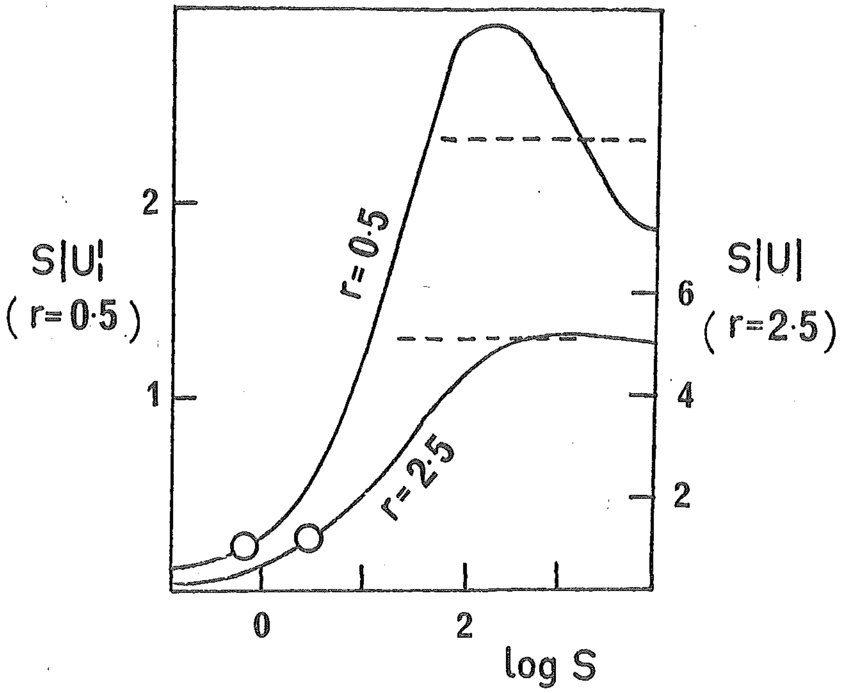

FIGURE (11.1)

This reproduces Yanowitch's wave solutions for gravity waves propagating in a viscous atmosphere. U is the horizontal component of the perturbation velocity while s is a height parameter varying as $\rho_0^{\frac{1}{2}}$. The dashed horizontal lines indicate the asymptotic behaviour of an upgoing gravity wave in an inviscid medium. The open circles give the reflection level calculated from the eikonal equations in the Boussinesq limit.

r = 2.5. Equation (10.14) gives s = -0.17 and 0.46 respectively. The eikonal equations have thus predicted a reflection level within half an atmospheric scale height of that càlculated from the full wave solutions.

Approximate expressions for the reflection coefficient may be found. Writing

$$I = \int_{-\infty}^{\infty} \tfrac{1}{2} \frac{\mu}{\rho_0} k^2 dt \tag{11.15}$$

where t = 0 at $k_z = 0$, then the reflection coefficient is simply,

$$|K_R| = \exp(-I) \tag{11.16}$$

Now, using equation (5.8), make a change of variable,

$$\begin{aligned} I &= \int_{-a}^{a} \tfrac{1}{2} \frac{\mu}{\rho_0} k^2 \frac{dt}{dk_z} \cdot dk_z \\ &= \int_{-a}^{a} \frac{2\sigma\rho_0}{\mu k^2} dk_z \end{aligned} \tag{11.17}$$

where $a = k_x\left(\frac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}}$ is the vertical wave number in the inviscid asymptote. But ρ_0 depends upon the reduced height z and we may use equation (1117) to eliminate this dependence.

$$I = \int_{-a}^{a} \frac{dk_z}{\left(\frac{N^2 k_x^2}{\sigma^2 k^2} - 1\right)^{\frac{1}{2}}} \tag{11.18}$$

Further writing

$$k_z^{\;2} = k_x^{\;2}\left(\frac{N^2}{\sigma^2} - 1\right)\sin^2\theta$$

$$I = \int_{-\pi/2}^{\pi/2} k_x\left[1 + \left(\frac{N^2}{\sigma^2} - 1\right)\sin^2\theta\right]^{\frac{1}{2}} d\theta$$

$$\approx k_x\left(\frac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}} ; \qquad \frac{N^2}{\sigma^2} - 1 \;\; \text{large}$$

$$\approx \pi k_x\left[1 + \tfrac{1}{4}\left(\frac{N^2}{\sigma^2} - 1\right)\right]; \qquad \frac{N^2}{\sigma^2} - 1 \;\; \text{small} \tag{11.19}$$

The reflection coefficient is approximately,

$$|K_R| \approx \exp\left[-\frac{2\pi H}{\lambda_z}\right] ; \qquad \frac{N^2}{\sigma^2} - 1 \;\; \text{large}$$

$$\approx e^{-\frac{2\pi}{\lambda_x}} \exp\left[-\frac{\pi^2 H\lambda_x}{4\lambda_z^{\;2}}\right]; \qquad \frac{N^2}{\sigma^2} - 1 \;\; \text{small} \tag{11.20}$$

The integral I includes the point $k_z = 0$, and at this point the WKB approximation is invalid. The reflection coefficient is thus at variance with the expression derived by Yanowitch,

$$|K_R| = \exp\left[-\frac{2\pi^2 H}{\lambda_z}\right] \tag{11.21}$$

Greater accuracy is apparent for large vertical wave numbers where the WKB approximation is valid within a level closer to the reflection point.

In the case when the effects of stratification are not neglected, the dispersion equation becomes,

$$k^2\sigma^2 - N^2k_x^{\;2} + \frac{i\sigma\mu}{\rho_0}k^4 + i\beta k_z\sigma^2 = 0 \tag{11.22}$$

As shown in a previous example, the effects of stratification depend primarily on the position of the wave packet in the atmosphere. Viscous effects on the other hand depend upon the 'time of flight' and only when the velocity of the packet is constant are these effects simply related. Thus it is not surprising that equation (11.9) does not factor. Effects attributed to viscosity and stratification are inter-related and no simple factorization will separate them.

We conclude that the eikonal method of ray tracing gives direct information when the dispersion equation is derived from primitive equations whose dependent variables are field variables. The method may be extended to include dissipative effects or the effects of stratification when the field variables are replaced by their approximate algebraic equivalents, that is by the application of a transformation which reduces the dispersion relation to a real form. For the cases examined, the algebraic approximation gives a ray trace which preserves the essential wave behaviour but is subject to some arithmetic error.

CHAPTER 12

WAVES IN A MEDIUM WITH FINITE SCALAR CONDUCTIVITY AND WITH PEDERSEN CONDUCTIVITY

To this point we have considered wave behaviour in a lossless medium. We now consider how waves behave when electrical dissipation is present, first in the simple case where the conductivity is finite and scalar. We continue to neglect the displacement current and consider only an incompressible medium. The hydrodynamic equations are unaltered but for the sake of completeness will be repeated here.

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_0}\nabla p - g + \mu\underline{J}\times\underline{\mathcal{H}} \tag{12.1}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0 \tag{12.2}$$

$$\nabla\cdot\underline{u} = 0 \tag{12.3}$$

$$\underline{J} = \nabla\times\underline{\mathcal{H}} \tag{12.4}$$

where $\underline{\mathcal{H}}$ represents the total magnetic field $\underline{H} + \underline{h}$.

The perturbation equivalent of these equations is,

$$\frac{\partial_2\underline{u}}{\partial t^2} + N^2(\underline{u}\cdot\hat{k})\hat{k} = -\frac{1}{\rho_0}\nabla\frac{\partial p}{\partial t} + \mu(\nabla\times\frac{\partial\underline{h}}{\partial t})\times\underline{H} \tag{12.5}$$

On the other hand, Maxwell's equations become, Chandrasekhar (1961, p149)

$$\frac{\partial \underline{\mathscr{H}}}{\partial t} + \nabla \times \left(\frac{1}{\eta}\nabla \times \underline{\mathscr{H}} - \underline{u} \times \underline{\mathscr{H}}\right) = 0 \tag{12.6}$$

where η is the conductivity of the medium.
Equation (6.11) has a perturbation form as follows,

$$\frac{\partial \underline{h}}{\partial t} + \frac{1}{\eta} \nabla \times (\nabla \times \underline{h}) = \nabla \times (\underline{u} \times \underline{H}) \tag{12.7}$$

Now writing,

$$\nabla \times (\nabla \times \underline{h}) = \nabla(\nabla\cdot\underline{h}) - \nabla^2\underline{h} \tag{12.8}$$

then we may set $\nabla\cdot\underline{h} = 0$ since space charges have been prohibited. Define an operator

$$L = \frac{\partial}{\partial t} - \frac{1}{\eta} \nabla^2 \tag{12.9}$$

whence

$$L\,\underline{h} = \nabla \times (\underline{u} \times \underline{H}) \tag{12.10}$$

At this point we may proceed as in Chapter 3; the derivation of the differential equation describing the vertical component of the perturbation velocity runs parallel to that given. Equation (12.5) with $\underline{h}$ eliminated contains an additional differential operator which increases the order of the derivatives with respect to z by two. Two equations result, one of the sixth order and the other of the fourth. However, if we confine our attention to the zero order WKB approximation, then L becomes an algebraic operator,

$$\underline{h} = \frac{1}{L} \nabla \times (\underline{u} \times \underline{H}) \tag{12.11}$$

and

$$\frac{\partial \underline{h}}{\partial t} = \frac{1}{1+i(k^2/\eta\sigma)} \nabla \times (\underline{u} \times \underline{H}) \tag{12.12}$$

To this order of approximation, $c_a^{\ 2}$ is replaced by $c_a^{\ 2}/(1+ik^2/\eta\sigma)$ and the dispersion equation becomes,

$$k^2\left[\sigma^2 - \frac{c_a^{\ 2}k_s^{\ 2}}{1+ik^2/\eta\sigma}\right] = k_h^{\ 2}N^2 \tag{12.13}$$

Care is needed when interpreting equation (12.13). It is of the third order in σ and no simple transformation of σ into real and imaginary parts exists. This results from having raised the order of the differential equation. However, we may approximate for each of the limits corresponding to high and low conductivities. For $k^2/\eta\sigma < 1$ then,

$$k^2\left[\left(\sigma + \tfrac{1}{2}i\,\frac{c_a^{\ 2}k_s^{\ 2}}{\sigma^2}\cdot\frac{k^2}{\eta\sigma}\right)^2 - c_a^{\ 2}k_s^{\ 2}\left(1 - \tfrac{1}{4}\,\frac{c_a^{\ 2}k_s^{\ 2}}{\sigma^2}\cdot\frac{k^4}{\eta^2\sigma^2}\right)\right]$$

$$= k_h^{\ 2}N^2 \tag{12.14}$$

This equation may be interpreted as a dissipating wave by virtue of the complex frequency; it has similar behaviour to that established when dissipation is absent. The effective Alfvén velocity however has been lowered, resulting in an increase in the level of mode conversion. The profile of η must be so chosen that $k^2/\eta\sigma < 1$ for all z. Since k^2 behaves as exp(-z) in the Alfven asymptote then η must increase with height to at least this order. If $k^2/\eta\sigma > 1$ then equation (12.13) approaches a form quadratic in σ and the dispersion relation admits the approximation,

$$k^2\left[\left(\sigma + \tfrac{1}{2}i\,\frac{c_a{}^2 k_s{}^2}{k^2}\,\eta\right)^2 + \tfrac{1}{4}\,\frac{c_a{}^4 k_s{}^4}{k^4}\,\eta^2\right] = k_h{}^2 N^2 \qquad (12.15)$$

Continuity between these two asymptotes can be expected. If we replace σ in equation (12.13) by $i\mu$, then a cubic in μ results having all coefficients real. This cubic has either three real or one real plus complex conjugate roots. In general all roots are continuous functions of z. Thus equation (12.13) has at least one pure imaginary root for σ representing an evanescent mode. The other two roots are continuous and may be pure imaginary or complex. Matching of the complex roots for each asymptote is thus always possible.

We now consider waves in a lightly ionized atmosphere where the magnetic pressure far exceeds the gas pressure of the medium. Such conditions pertain to the ionosphere. We may imagine that the ionized fraction of this atmosphere is rigidly fixed to an immovable magnetic field line. Motions of the neutral gas perpendicular to the field suffer degradation by collisions with the ionized fraction while the longitudinal component of the motion is unaffected. Perturbations of the magnetic field are thus neglected. Gershman and Grigar'yev (1965) have given relevant approximations pertinent to this example and show that Pedersen conductivity only need be considered. We further constrain conditions by restricting discussion to the Boussinesq limit, we also retain the assumption of an incompressible fluid. The equations of motion become,

$$\frac{\partial^2 \underline{u}}{\partial t^2} + N^2(\underline{u}\cdot\hat{k})\hat{k} = -\frac{1}{\rho_0}\nabla\frac{\partial p}{\partial t} + \gamma\left(\frac{\partial \underline{u}}{\partial t}\times\hat{s}\right)\times\hat{s} \tag{12.16}$$

$$\nabla\cdot\underline{u} = 0 \tag{12.17}$$

where $\hat{s}$ is the unit vector directed along the magnetic field line and

$$\gamma = \frac{\mu H^2}{\rho_0}\eta_p \tag{12.18}$$

The variable γ thus characterizes the electrical properties of the atmosphere.

Employing the zero order WKB approximation as well as using the commutative properties of the differential operators implicit in the Boussinesq limit, then equations (12.16) and (12.17) reduce to their matrix equivalents,

$$\begin{bmatrix} -(\sigma^2+i\sigma\gamma\sin^2\theta) & 0 & i\sigma\gamma\sin\theta\cos\theta & -\frac{i\sigma k_x}{\rho_0} \\ 0 & -(\sigma^2+i\sigma\gamma) & 0 & -\frac{i\sigma k_y}{\rho_0} \\ i\sigma\gamma\sin\theta\cos\theta & 0 & N^2-(\sigma^2+i\sigma\gamma\cos^2\theta) & -\frac{i\sigma k_z}{\rho_0} \\ ik_x & ik_y & ik_z & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = 0 \tag{12.18}$$

The coordinate system is that used previously. The magnetic field lies in the X-Z plane and is inclined at an angle θ to the horizontal. The dispersion equation is simply the determinant of this matrix equation and is,

$$k^2\left[\sigma^2 + i\sigma\gamma\frac{k_s^2}{k^2}\right] = N^2\left[k_x^2 + k_y^2\,\frac{\sigma+i\gamma\sin^2\theta}{\sigma+i\gamma}\right] \tag{12.19}$$

This expression is the same as that derived for scalar conductivity except for an additional factor appearing as the coefficient of k_y^2. For vertical magnetic field or for motion confined to the meridional plane, this term vanishes and the dispersion equation becomes identical to equation (12.13). The interpretation of equation (12.19) is not the topic of this thesis and it would be impr dent to draw conclusions without support from wave solutions. However, curious wave behaviour is possible, particularly near the level defined by $\gamma = \sigma$. This may be of ionospheric significance.

A summary of the dispersion equations derived so far is;

Infinite conductivity,

$$k^2\left[\sigma^2 - c_a^2k_s^2\right] = k_h^2N^2 \tag{12.20}$$

Finite scalar conductivity,

$$k^2\left[\sigma^2 - \frac{c_a^2k_s^2}{1+\frac{ik^2}{\sigma\eta}}\right] = k_h^2N^2 \tag{12.21}$$

Pedersen conductivity, motion confined to the magnetic meridian,

$$k^2\left[\sigma^2 + i\sigma\gamma\frac{k_s^2}{k^2}\right] = k_h^2N^2 \tag{12.22}$$

For the sake of comparison, the dispersion equation of gravity waves in a viscous medium is also included,

$$k^2\left[\sigma^2 + \frac{i\sigma\mu}{\rho_0}k^2\right] = k_h^2N^2 \tag{12.23}$$

These equations show continuity dependent upon the parameter $k^2/\eta\sigma$. Clearly equation (12.21) becomes identical to equation (12.20) in the limit where η tends to infinity, while making $k^2/\eta\sigma > 1$ it approximates the form of equation (12.22). Alternatively equation (12.22), which represents wave propagation in a medium with resistive dissipation, is similar in form to the dispersion equation found when dissipation is by viscous forces alone. In any situation, the limiting form, represented by either equation (12.20) or equation (12.22), is chosen according to the parameter $k^2/\eta\sigma$. If this ratio is less than unity then reactive interaction with the magnetic field dominates. An up-going gravity wave dissipates but conversion to an Alfvén mode may still occur. For this ratio greater than unity, behaviour is analogous to that in a viscous medium. The detailed nature of the behaviour pattern will depend on how the conductivity η and also the ratio k^2/η varies with height. Subject to the qualifications already given, equation (12.20) and equation (12.23) may be used as a basis for ray tracing. With similar qualifications it may also be applied with some confidence to equation (12.21) and (12.22).

The variation of the vertical wave number with time gives some insight into wave behaviour. Using equation (5.7) to derive an expression for this from equation (12.14) gives,

$$\frac{dk_z}{dt} = \frac{\partial G}{\partial z} \bigg/ \frac{\partial G}{\partial \sigma}$$

$$= \frac{c_a^2 k_s^2}{2\sigma k^2} \left(1 - \frac{1}{4}\,\frac{c_a^2 k_s^2}{\sigma^2} \cdot \frac{k^4}{\eta^2\sigma^2}\right) \frac{d \log c_a^2}{dz} \qquad (12.24)$$

The terms contained within the bracket are here regarded as perturbation quantities only. To extrapolate behaviour beyond the bounds already set is illadvised. We note however that a zero of dk_z/dt exists and that dk_z/dt may change sign either before or after the conditions of mode conversion have been satisfied. Maximum wave dissipation occurs near the level of this zero which in turn depends on both wave and atmospheric parameters. This level does not necessarily coincide with the maximum height to which a wave packet will penetrate. A wide variety of behaviour is therefore possible.

The behaviour of k_z with time when the conductivity is low may be found from either equation (12.15) or equation (12.22). The relevant expression is,

$$\frac{dk_z}{dt} = \tfrac{1}{4}\gamma^2 \frac{k_s^4}{k^4} \frac{d \log \gamma^2}{dz} \geqslant 0 \qquad (12.25)$$

Behaviour is here analogous to that for a viscous atmosphere. The equivalent level of reflection is defined by $k_z = 0$. Albeit, the dissipation rates depend upon k_s; the level given by $k_s = 0$ is one of zero dissipation and if encountered by a wave packet will have a non-vanishing vertical component of group velocity. In the absence of other dissipative processes the wave may well be trapped. Equation (12.25) indicates that all higher derivatives of k_z with time also vanish. If $k_s = 0$ is not encountered on the flight path then a minimum k_s will occur and asymmetry introduced into the expressions, the maximum dissipation then occurs at a level below that defined by $k_z = 0$ on the alternate leg of the path.

CHAPTER 13

APPLICATION TO THE TERRESTRIAL AND SOLAR ATMOSPHERES

The ionosphere may be regarded as a binary medium consisting of a neutral and ionized fraction. In the lower ionosphere these fractions are collisionally well coupled while in the upper ionosphere collisional interaction within the plasma is ultimately more effective than between fractions. First consider that good coupling between components exists throughout the ionosphere. Figure (13.1) is abstracted from Chapman (1956) and plots the electrical conductivity as a function of altitude. Using these values of conductivity in equation (12.21) then for disturbances with frequency components near the Brunt frequency, only those having characteristic lengths exceeding 10^3 km does reactive interaction with the magnetic field become important and the fluid behave as if it were a perfect conductor. It is interesting to note that the partial reflection of atmospheric oscillations in the form of magnetic variations occurs near this limit. The present study adds little to the study of ionospheric perturbations.

Alternatively there seems little point in pursuing a search for the magnetic effects associated with other than this class of travelling disturbance as dissipative processes act in such a way as to preclude coupling with the electromagnetic spectrum.

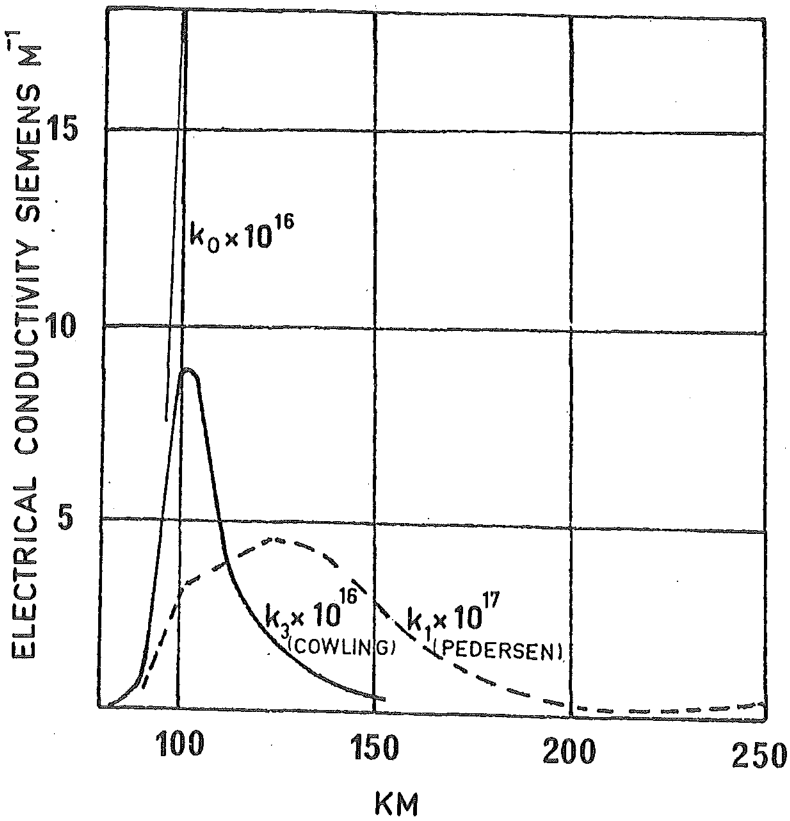

FIGURE 13.1

The electrical conductivity of the ionosphere. Values are abstracted from Chapman (1956) and refer to conditions over White Sands, New Mexico.

Waves might exist in the limit where the ionized fraction is, to the first approximation, collisionally decoupled from the neutral atmosphere. However, the generation of such waves is problematic since energy, if introduced from below by coupling with the natural component, is severely dissipated. Put crudely, only above that level where the ion-neutral collisional frequency is comparable to the wave frequency may these freely propagate.

The derivation of the dispersion equation as given in Chapter 11 has implied that theinclusion of conductive dissipation raises the differential equation describing the wave behaviour from the fourth order to one of the sixth order. To extrapolate the properties of waves across this increase of order is dubious. This is particularly so when the checks provided by ray tracing methods lose validity when high dissipation is present. If a linearized theory is able to explain the various anomalies in the observed behaviour of travelling disturbances that exist in the ionosphere then it must include the effects of compressibility, viscosity and anisotropic conductivity. It would thus be a high order theory and beyond the scope of this thesis.

We now consider the possible application to the solar chromosphere. It is commonly believed that mechanical waves, generated by the turbulence in the photosphere traverse this region and deposit their energy in the solar corona. The solar corona, in apparent defiance of the second law of thermodynamics is maintained at a temperature some thousandfold that of the underlying chromosphere. The detailed mechanisms for the generation, propagation and absorption of this energy have been hotly debated in recent years. The current ideas are crystallized in a paper by Lighthill (1967) and the discussion subsequent to this paper gives insight into the difficulties of constructing a theory which satisfactorily explains the energy balance of the sun's upper regions.

Under quiet solar conditions, the region of transition between the solar photosphere and the chromosphere is one of surprising tranquility. Convective motions have estimated velocities of between five and twenty per cent that of the local velocity of sound, a fraction of about the same order as that observed for meteorological winds. The scale size of motions as inferred from solar granulations is about 800 km while their mean life is perhaps 8 minutes, Minnerat (1953), Schwartzchild (1961). Osterbrock (1961) supports the view that the outflow of energy from the photosphere may be transported by sound waves generated by photospheric convection. These waves may ultimately be converted to shocks. By invoking such a non-linear process it is possible to ensure a net upflow of energy thereby maintaining the corona at its high temperature, as well as supplying the energy ultimately lost by chromospheric radiation. However, Lighthill (1967) states, '... it appears marginally possible, but by no means certain, that sound so generated can quantitatively be sufficient to account for the chromospheric heating'. Further, Ulmschneider (1970) notes that no travelling shocks have been observed in the solar chromosphere and accounts for this failure of observation by postulating shock formation from turbulence having a period of about 10 sec.

We now consider the generation of gravity waves. Turbulent regions do not generate gravity waves since stable stratification is absent and gravity waves do not propagate within these regions. However the generation of waves is possible near the boundary of an unstable region. Stein (1967) has shown that the generation of gravity waves from the quasi-stable regions near the photosphere is more efficient than that of sound waves. Whereas the acoustic power output varies as M^5, M being the Mach number, the upwards gravity flux depends upon the fifth power of the scale size expressed in units of the atmospheric scale height. There appears to be a sufficient power generation to balance the $10^{10}\mu Jm^{-2}s^{-1}$ emitted from the chromosphere as line radiation and the $10^{8}\mu Jm^{-2}s^{-1}$ needed to maintain the coronal energy balance. Since the extension of the layers of turbulent penetration are unknown, we cannot estimate this flux with any reliability. Gravity waves must still be considered as a possible mechanism for the transport of energy.

Table (13.1) gives physical parameters applicable to the upper photosphere and the chromosphere. These are taken from Osterbrock (1961). If gravity waves have scale sizes corresponding to those of the solar granulations then dissipative effects may be neglected and the medium behaves as if it were of infinite conductivity. For setting $\sigma \sim 0\ 10^{-2}s^{-1}$ and $k \sim 0\ 10^{-5}m^{-1}$, values corresponding to solar granulating, then at the base of the chromosphere,

$$\frac{\mu k^2}{\rho_0 \sigma} = -\ 10^{-8}\ ; \qquad \frac{k^2}{\sigma\eta} = 0\ 10^{-4} \qquad (13.1)$$

h km	log N_e	log N_H	V_s	V_A		$\frac{1}{\eta}$	$\frac{\mu}{\rho_0}$
				B = .2 mT	B = 5 mT		
-400	20.8	23.2	8.2	0.0082	0.20	1.3^{3}	1.3^{-1}
-300	19.5	23.0	6.7	0.010	0.25	1.4^{4}	1.6^{-1}
-200	18.9	22.7	6.5	0.015	0.38	2.2^{4}	3.1^{-1}
-100	18.5	22.2	6.4	0.025	0.62	2.4^{4}	8.3^{-1}
0	18.0	21.7	6.3	0.041	1.0	2.3^{4}	2.6
250	17.4	21.2	6.4	0.092	2.3	1.8^{4}	8.9
500	17.0	20.5	6.5	0.20	5.0	1.3^{4}	4.6
750	16.8	10.9	6.6	0.41	10.2	6.2^{4}	1.8^{2}
1000	16.7	10.3	6.7	0.82	20.0	2.9^{4}	7.6^{2}

TABLE 13.1 Physical parameters in the upper photosphere and lower chromosphere of the sun (after Osterbrock, 1961). The parameters are defined as follows,

h height measured from the base of the chromosphere

N_e electron density (m^{-3})

N_H density of neutral hydrogen (m^{-3})

V_s the velocity of sound (km s^{-1})

V_A the Alfven velocity (km s^{-1})

$\frac{1}{\eta}$ resistivity in siemens m^{-1}

$\frac{\mu}{\rho_0}$ kinematic viscosity, $m^2 s^{-1}$.

Gravity waves propagating at this level suffer little dissipation and as a first approximation we may apply the theory outlined in Part I. In the vicinity of the solar granulations the magnetic field is small; we take the Alfven velocity corresponding to the smaller field to represent conditions near the granulations. Using these values to find the level of mode conversion, defined where $c_a^2 k_x^2 = \sigma^2$, then this occurs near the level where $c_a \approx 1$ km/sec or at $h \approx 1000$ km. This level is the approximate level of transition from the chromosphere to the solar corona.

Table (13.1) shows that conductive dissipation is more effective than viscous dissipation. We neglect viscosity. Moreover the conductivity is ostensibly constant. Following the motion of a wave packet then the wave number increases until dissipation dominates. However, we must increase the wave number by about 10^2 before the wave is extinguished. Reference to Figure (4.1) indicates that for the wavelengths and frequencies under consideration this level is near the emitting level when the magnetic field is vertical and near the level of mode conversion for a horizontal magnetic field. The energy deposition is consequently unlike that predicted for gravity waves uninfluenced by magnetic effects.

So far we have seen that gravity waves may well supply the energy necessary to heat the solar chromosphere and the corona and that the probable scale sizes of these waves are such that mode conversion is a likely physical process in these regions of the solar atmosphere. It is therefore possible that the solar atmosphere is heated by the dissipation of waves which initially propagated upwards as gravity waves;

the dissipation occurring either in the Alfvén mode by the processes already outlined, or as has been shown possible in Chapter (8), the conversion to a fast Alfvén mode and subsequent shock formation. As dictated by the law of parsimony we consider first the linear processes. So little is known about the generation of gravity waves and how the energy is distributed that only a brief description is possible.

It is crucial to this hypothesis that the specific energy input by the dissipation of waves should increase with altitude since failure to satisfy this condition would certainly lead to atmospheric instability. As a crude model we consider the case where only those waves with $|\underline{k}|$ and its horizontal component $|\underline{k}_h|$ being constant may propagate; the horizontal wave vector $\underline{k}_h$ is assumed to be uniformly distributed showing no preferential orientation with respect to the magnetic meridian. We may approximate the propagating wave as follows: a gravity wave propagates upwards with a constant wave vector $\underline{k}$ until its group velocity given by equation (4.15) becomes zero, mode conversion then occurs and the downward propagating Alfvén wave has a vertical component of wave number which increases exponentially. Wave absorption is a maximum at some fixed increment of reduced height below the level of mode conversion since this depends only on $|\underline{k}|$. The heat energy generated by wave absorption is distributed in a way that is related only to the distribution of the level of mode conversion by some functional transformation as yet unknown. We take the distribution of the level of mode conversion to indicate the functional form of the heat input.

Consider the simplest possible model, that is when the magnetic field is horizontal. Critical behaviour is at that level defined by the expression

$$c_a^{\,2} k_x^{\,2} = \sigma^2 . \tag{13.2}$$

Now for waves uniformly distributed with the asmuthal angle ϕ, the distribution of the critical level is found from equation (13.2) by differentiation.

$$\frac{1}{c_a^{\,2}} \frac{dc_a^{\,2}}{dz} \cos^2\phi = 2 \sin\phi \cos\phi \frac{d\phi}{dz} . \tag{13.3}$$

But for constant magnetic field,

$$\frac{1}{c_a^{\,2}} \frac{dc_a^{\,2}}{dz} = \rho_0 \frac{d}{dz} \left(\frac{1}{\rho_0}\right) = -\frac{1}{\rho_0} \frac{d\rho_0}{dz} . \tag{13.4}$$

The specific heating Φ is proportional to dz/ρ_0 and is,

$$\Phi \propto \frac{c_a^{\,2}}{\left[-\dfrac{1}{\rho_0}\dfrac{d\rho_0}{dz}\right]} \tan\phi \, d\phi \tag{13.5}$$

subject to the condition

$$c_a^{\,2} k_h^{\,2} \cos^2\phi = \sigma^2 . \tag{13.6}$$

This yields a final expression

$$\Phi \propto \frac{c_a^{\ 2}}{\left[-\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right]}\left[\frac{c_a^{\ 2}k_h^{\ 2}}{\sigma^2} - 1\right]^{\frac{1}{2}} \tag{13.7}$$

The specific heating rate therefore increases monotonically upwards and is of the required functional form.

So far it has been tacitly assumed that the energy is propagated at scale sizes close to those of the solar granulations. Stein (1967) has shown that this is not so. It is heavily weighted towards the longer wave lengths of the spectrum. However, an increase in wave length decreases the effective dissipation and increases the level of mode conversion. Since both of these depend upon $k_x^{\ 2}$, the height of maximum dissipation is substantially unaltered by this change in scale size. All wave lengths in the spectrum, but at a given frequency, tend to deposit their energy at a given level. The heat input function is thus, to a first approximation, a function easily separated into factors containing frequency and wave number and is a function which in general increases monotonically with altitude.

An increase in the ambient temperature decreases the Brunt frequency resulting in a decrease in the vertical wave number. This may ultimately lead to the reflection of gravity waves. However, the effects of conduction are such that the vertical wave number increases and the wave is converted to an Alfvén wave. It is not immediately clear which process will dominate. To resolve this we consider an atmosphere where the Brunt frequency N varies with height. The wave number k is kept large so that the WKB approximation applies. The wave behaviour is determined by the sign of dk_z/dt as we follow a wave packet propagating through this atmosphere, and this in turn depends upon the sign of G/ z in the relevant dispersion equation. Applying this criterion to equation (12.20) then,

$$\frac{\partial G}{\partial z} = -k^2 \left[c_a{}^2 k_s{}^2 \left(\frac{d \log c_a{}^2}{dz} - \frac{d \log N^2}{dz} \right) + \sigma^2 \frac{d \log N^2}{dz} \right], \tag{13.8}$$

where $\frac{d \log c_a{}^2}{dz} = \frac{1}{H} > 0$

and $\frac{d \log N^2}{dz} \approx \frac{d}{dz}(\frac{1}{H}) < 0.$

k_z thus decreases for a temperature increase whose scale is at least that of the atmospheric scale height. Mode conversion dominates wave reflection in this case.

We have speculated on the behaviour of gravity waves in a solar like atmosphere and have shown that these waves appear to have the necessary attributes for transporting energy to the solar corona. Wave properties have been extrapolated from a Boussinesq approximation in which derivatives of ambient parameters are neglected. Dissipation is considered to be small. It has already been shown that low order approximations can yield entirely misleading results and that full wave analysis is the only sure way of checking conjectures. Solutions have been found without reference to the way in which the equilibrium wave system might be established or even the stability of any intermediate wave system which might reach this final state. Further, it is dangerous to infer the behaviour of transient waves from their final equilibrium condition. These factors and many others must be considered before a case for the coronal heating could be regarded as more than speculative. However, some delightful possibilities exist and these must await more elegant treatment than that given here.

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APPENDIX I

GLOSSARY OF COMMONLY USED SYMBOLS

$\underline{B}$	magnetic induction
c_a	the local Alfvén velocity
c_0	the velocity of sound
$\underline{E}$	the induced electric field
G_u	complex amplitude of a gravity wave with an upwards component of energy flux,
G_d	complex amplitude of a downgoing gravity wave
$\underline{H}$	strength of the ambient magnetic field
$\underline{h}$	wave perturbation of the magnetic field
$\mathscr{H}$	the scale of the density stratification of the medium
$\mathscr{I}$	imaginary part of a complex quantity
$\underline{J}$	current density vector
$\underline{k}$	wave vector with components (k_x, k_y, k_z)
$\underline{k}_h$	horizontal component of $\underline{k}$,
$\mathscr{L}$	the Lorentz force
l	ratio of Brunt frequency to wave frequency
M_u	complex amplitude of an upgoing Alfvén wave
M_d	complex amplitude of a downgoing Alfvén wave
N	the Brunt frequency
p	pressure perturbation
Re	the real part of a complex quantity
$\hat{s}$	unit vector directed along the ambient magnetic field
s	distance measured in the direction of $\hat{s}$
u	the fluid's perturbation velocity having components (u,v,w)

$\underline{v}_g$	group velocity of the wave, components (v_{gx}, v_{gy}, v_{gz})
$\underline{X}$	the position vector (x,y,z) normalized by the scale of the density stratification
η	isotropic electrical conductivity
η_p	Pedersen conductivity
θ	angle of inclination of the ambient magnetic field measured as an angle of rotation from the X-axis in the X-Z plane
λ	normalized density of the stratified fluid
μ	magnetic permeability
ρ	the density perturbation
ρ_0	density of the unperturbed fluid
ρ_{00}	density of the unperturbed fluid at the reference level
σ	wave frequency
σ_a	the Alfvén frequency, $\sigma_a^2 = c_a^2 k_x^2$
ϕ	angle subtended by the vector $\underline{k}_h$ to the X-Z plane
ϕ_0	the phase integral
ω	the vertical component of vorticity

APPENDIX II

APPROXIMATE ROOTS TO THE DISPERSION EQUATION AT SMALL σ/σ_a

The equation

$$\left(\frac{k_z^2}{k_x^2}+\frac{k_h^2}{k_x^2}\right)\left[\left(\frac{k_z}{k_x}\sin\theta+\cos\theta\right)^2-\frac{\sigma^2}{\sigma_a^2}\right]+\frac{k_h^2}{k_x^2}\cdot\frac{N^2}{\sigma^2}\cdot\frac{\sigma^2}{\sigma_a^2} = 0 \qquad \text{(A.II.1)}$$

has roots

$$k_z = \pm\, ik_h,\ -k_x\cot\theta,\ -k_x\cot\theta \qquad \text{(A.II.2)}$$

when σ/σ_a is small. Higher approximations can be found by expanding these roots as a power series in σ/σ_a about $\sigma/\sigma_a = 0$. Only the first term of this expansion is required.

Inserting

$$k_z = \pm\, ik_h(1+\varepsilon) \qquad \text{(A.II.3)}$$

into A.II.1 gives,

$$(\not{\varepsilon}^2+2\varepsilon)\left[\left(\cos\theta\pm\frac{ik_h}{k_x}(1+\varepsilon)\sin\theta\right)^2-\frac{\sigma^2}{\sigma_a^2}\right] = \frac{N^2}{\sigma^2}\cdot\frac{\sigma^2}{\sigma_a^2} \qquad \text{(A.II.4)}$$

Retaining only the first power of ε and of σ^2/σ_a^2 then

$$\varepsilon \approx \frac{\frac{1}{2}\frac{N^2}{\sigma^2}\cdot\frac{\sigma^2}{\sigma_a^2}}{\left(\cos\theta\pm\frac{ik_h}{k_x}\sin\theta\right)^2} \qquad \text{(A.II.5)}$$

Thus

$$k_z \approx \pm ik_h \left[1 + \frac{\frac{1}{2}\frac{N^2}{\sigma^2} \cdot \frac{{}^2}{\sigma_a^2}}{\left(\cos\theta \pm \frac{ik_h}{k_x} \sin\theta \right)^2} \right] \qquad \text{(A.II.6)}$$

The expansion for the second expression can be found by writing

$$k_z = - k_x \cot\theta \, (1+\varepsilon). \qquad \text{(A.II.7)}$$

This gives

$$\left[(1+\varepsilon)^2 \cot^2\theta + \frac{k_h^2}{k_x^2} \right] \left[\varepsilon^2 \cot^2\theta - \frac{\sigma^2}{\sigma_a^2} \right] = - \frac{k_h^2}{k_x^2} \cdot \frac{N^2}{\sigma^2} \cdot \frac{\sigma^2}{\sigma_a^2}$$

(A.II.8)

and by collecting terms

$$\left[(1+\varepsilon)^2 \cot^2\theta + \frac{k_h^2}{k_x^2} \right] \varepsilon^2 \cot^2\theta = \frac{\sigma^2}{\sigma_a^2} \left[(1+\varepsilon)^2 \cot^2\theta - \left(\frac{N^2}{\sigma^2} - 1 \right) \frac{k_h^2}{k_x^2} \right]$$

(A.II.9)

To the lowest order of small quantities then

$$\varepsilon \cot\theta = \pm \frac{\sigma}{\sigma_a} \left[\frac{\cot^2 - \left(\frac{N^2}{\sigma^2} - 1 \right) \frac{k_h^2}{k_x^2}}{\cot^2\theta + \frac{k_h^2}{k_x^2}} \right]^{\frac{1}{2}} \qquad \text{(A.II.10)}$$

The final approximation is therefore

$$k_z = -k_x \left\{ \cot\theta \pm \frac{\sigma}{\sigma_a} \left[\frac{\cot^2\theta - \left(\frac{N^2}{\sigma^2} - 1 \right) \frac{k_h^2}{k_x^2}}{\cot^2\theta + \frac{k_h^2}{k_x^2}} \right]^{\frac{1}{2}} \right\}$$

(A.II.11)

APPENDIX III

NUMERICAL INTEGRATION OF THE BIHARMONIC WAVE EQUATION

Given a fourth order equation of the form

$$\frac{d_4 X}{dz^4} + A_3 \frac{d_3 X}{dz^3} + A_2 \frac{d_2 X}{dz^2} + A_1 \frac{dX}{dz} + A_0 X = 0 \qquad \text{(A.III.1)}$$

where both X and A are complex quantities, write

$$\begin{aligned} A_\gamma &= \alpha_\gamma + i\beta_\gamma \\ X &= u + iv \end{aligned} \qquad \text{(A.III.2)}$$

We wish to find the solution of this equation by numerical methods. We treat it as a system of coupled first order equations. Let

$$[Y] = \begin{bmatrix} u \\ v \\ u' \\ v' \\ u'' \\ v'' \\ u''' \\ v''' \end{bmatrix} \qquad \text{(A.III.3)}$$

then

$$\frac{d[Y]}{dZ} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\alpha_0 & \beta_0 & -\alpha_1 & \beta_1 & -\alpha_2 & \beta_2 & -\alpha_3 & \beta_3 \\ -\beta_0 & -\alpha_0 & -\beta_1 & -\alpha_1 & -\beta_2 & -\alpha_2 & -\beta_3 & -\alpha_3 \end{bmatrix} [Y] \tag{A.III.4}$$

Since,

$$\frac{d}{dz} v'' = \frac{d^4 v}{dz^4} \quad \text{etc.}$$

Any standard scientific subroutine which solves a set of coupled first order equations can therefore be used. For $\theta = 90^\circ$ the IBM package HPCG (Hamming's predictor-corrector method) was used while for $\theta \neq 90^\circ$ RKGS (a Runge-Kutta method) was used. The HPCG package would not initialize for $\theta \neq 90^\circ$ and no reason was found for this. It was found sufficient to sample the solution about fifty times within a 'magnetic' wave length. Additional checks were carried out. Solutions were commenced at different values of z and checked by reverse integration.

APPENDIX IV

SAMPLE OF NUMERICAL SOLUTION

This gives the programme listings for wave computations when $\theta \neq 0^\circ$ or 90°. The programme requires the IBM scientific subroutine packages RKGS and POLRT. These are not listed here. The subroutine CDSMSL is a single precision version of that coded by Dr H.A. von Biel.

The sample output is self-explanatory, except perhaps for the penultimate table. This table gives the amplitudes of the equivalent wave system scaled by exp(-0.75z) for the magnetic components and by exp(-0.5z) for gravity components. As with other tables the columns are tabulated according to increasing values of vertical wave number. Thus column (1) gives values corresponding to a downgoing magnetic wave, column (2) that for an upgoing gravity wave, column (3) a downgoing gravity wave and column (4) refers to an upgoing magnetic wave. The first four columns refer to the solution S_1 while the remaining four refer to S_2.

```
      COMMON BOUS,BRUNT,WNX,WNY,THETA,XTOP,XBOT,BACK
      COMMON A(104,90),IPT(13),ICOUNT,DELT,NMAX,IFST
      EXTERNAL OUT,FCT
      REAL AUX(200),Y(8),DERY(8),PRMT(5),TITLE(13)
      REAL G(8),DUM(100)
      REAL W(4)
      COMPLEX DER(4),U(2)
      COMPLEX COM(4,5)
      COMPLEX SOL1(4),SOL2(4),CVAR
C                                                                        C
C************************************************************************C
C                                                                        C
C     POINTER DEFINITIONS                                                C
C        IPT(1)     SOLUTION FOR U=-I*KX*COT(THETA)                      C
C        IPT(2)     SOLUTION FOR U=KX                                    C
C        IPT(3)     DISPERSION VALUES                                    C
C        IPT(4)     PHASE INTEGRAL                                       C
C        IPT(5)     WKB DECOMPOSITICN OF SOLUTION 1                      C
C        IPT(6)     WKB DECOMPOSITION OF SOLUTION 2                      C
C        IPT(7)     AMPLITUDE AND PHASE OF SOLUTION 1                    C
C        IPT(8)     AMPLITUDE AND PHASE OF SOLUTION 2                    C
C        IPT(13)    REDUCED HEIGHT,FLUX1,FLUX2                           C
C                                                                        C
C************************************************************************C
C                                                                        C
    2 READ(5,551,ERR=3,END=3) TITLE

  551 FORMAT(13A6)
      READ(5,550,ERR=3,END=3) BOUS,BRUNT,WNX,WNY,THETA,XBOT,XTOP
  550 FORMAT(8G10.4)
      IF(BOUS.EQ.0.0) WRITE(6,601)

      IF(BOUS.NE.0.0) WRITE(6,602)
  601 FORMAT('1',T12,'***MODELLED IN THE BOUSSINESQ APPROXIMATION***')
  602 FORMAT('1',T12,'***MODELLED FOR AN INCOMPRESSIBLE FLUID***')
      WRITE(6,605) TITLE
  605 FORMAT('0',13A6)
      WRITE(6,603) BRUNT,WNX,WNY,THETA
  603 FORMAT('OTHE RATIO OF BRUNT FREQUENCY TO WAVE FREQUENCY ',1P(G10.4
     C)/' THE RATIO OF MERIDIONAL SCALE TO SCALE HEIGHT ',1P(G10.4)/
     C ' RATIO OF TRANSVERSE SCALE TO SCALE HEIGHT ',1P(G10.4)/
     C ' THE INCLINATION OF THE MAGNETIC FIELD ',1P(G10.4))
      WRITE(6,604) XBOT,XTOP
  604 FORMAT(' THE INTEGRATION COMMENCES AT ',1P(G10.4),'AND TERMINATES
```

```
C AT ',1P(G10.4))
      BACK=1.0
      X=XBOT
      IF(XTOP.GT.XBOT) GOTO 9
      BACK=-1.0
      XTOP=-XTOP
      XBOT=-XBOT
    9 CONTINUE
      DO 11 I=1,13
   11 IPT(I)=(I-1)*8+1
      THETA=THETA/57.2957795
      S=SIN(THETA)
      C=COS(THETA)
      WNH=SQRT(WNX*WNX+WNY*WNY)
      U(1)=CMPLX(0.0,WNX*C/S)
      U(2)=CMPLX(WNH,0.0)
      DELT=0.1
      IFST=0
    4 IFST=IFST+1
      ICOUNT=1
      CALL SERIES(X,BRUNT,WNX,WNY,C,S,U(IFST),1000,1.0E-6,DER)
      DO 10 I=1,4
      Y(2*I-1)=REAL(DER(I))
   10 Y(2*I)=AIMAG(DER(I))
      CALL FFCT(X,Y,DERY)
      CALL       FOUT(X,Y,DERY,IHLF,NDIM,PRMT)
      Y(3)=Y(3)*BACK
      Y(4)=Y(4)*BACK
      Y(7)=Y(7)*BACK
      Y(8)=Y(8)*BACK
      DO 1 I=1,8
    1 DERY(I)=0.125
      PRMT(1)= XBOT
      PRMT(2)= XTOP
      PRMT(3)=(XTOP-XBOT)/800.0
      PRMT(4)=0.0001
      PRMT(5)=0.0
      CALL RKGS(PRMT,Y,DERY,8,IHLF,FCT,OUT,AUX)
      IF(IFST.LT.2) GOTO 4
      ICOUNT=ICOUNT-1
      X2=XBOT
      X1=X2-DELT
      CALL QUAD(BOUS,BRUNT,WNX,WNY,X,DUM,S,C)
      DO 102 I=0,7
  102 A(IPT(3)+I,1)=DUM(I+1)
      III=ICOUNT
      DO 1000 ICOUNT=2,III
      X1=X1+DELT
      X2=X2+DELT
      DO 201 J=1,8
```

```
 201 G(J)=0.0
     DO 202 J=0,10
     CALL QUAD(BOUS,BRUNT,WNX,WNY,(X1+J*DELT/10.0)*BACK,DUM,S,C)
     F=0.1*DELT*BACK
     IF(J.EQ.0.OR.J.EQ.10) F=0.05*DELT*BACK
     DO 202 K=1,8
 202 G(K)=G(K)+DUM(K)*F
     DO 203 J=0,7
     A(IPT(3)+J,ICOUNT)=DUM(J+1)
 203 A(IPT(4)+J,ICOUNT)=G(J+1)+A(IPT(4)+J,ICOUNT-1)
1000 CONTINUE
     ICOUNT=III
     WRITE(6,650)
 650 FORMAT('0',T10,'***THE INITIAL SOLUTION***'/
    C           T10,'F(X) AND ITS DERIVATIVES FOR U=I*KX*COT(THETA)'/
    C           T10,'THE FINAL COLUMN TABULATES THE FLUX')
     WRITE(6,658)
 658 FORMAT('0')
     WRITE(6,651) ((A(IPT(13),I),(A(IPT(1)+J,I),J=0,7),A(IPT(13)+1,I)),
    C             I=1,ICOUNT)
 651 FORMAT(0PF6.2,2X,1P9G13.6)
     WRITE(6,652)
 652 FORMAT('1'///)
     WRITE(6,653)
 653 FORMAT('0',T10,'***THE INITIAL SOLUTION***'/
    C           T10,'F(X) AND ITS DERIVATIVES FOR U=KH'/
    C           T10,'THE FINAL COLUMN TABULATES THE FLUX')
     WRITE(6,658)
     WRITE(6,651) ((A(IPT(13),I),(A(IPT(2)+J,I),J=0,7),A(IPT(13)+2,I)),
    C               I=1,ICOUNT)
     WRITE(6,652)
     WRITE(6,654)
 654 FORMAT(T10,'***REAL AND IMAGINARY PARTS OF THE VERTICAL WAVE NUMBE
    CR***')
     WRITE(6,658)
     WRITE(6,655) ((A(IPT(13),I),(A(IPT(3)+J,I),J=0,7)),I=1,ICOUNT)
 655 FORMAT(0PF6.2,2X,1P8G13.6)
     WRITE(6,652)
     WRITE(6,656)
 656 FORMAT(T10,'***REAL AND IMAGINARY PARTS OF THE PHASE INTEGRAL***')
     WRITE(6,658)
     WRITE(6,655) ((A(IPT(13),I),(A(IPT(4)+J,I),J=0,7)),I=1,ICOUNT)
     JJJ=III-3
     IC=0
 403 INT=0
 400 INT=INT+1
     IP=IC*4
     F=-0.75*DELT*BACK
     H=-0.5*DELT*BACK
     DO 402 K=1,JJJ
     DO 401 I=1,4
     N=K+I-1
```

```
      FACT=5.5-I
      DO 404 L=1,4
      FRIGIT=1.0
      XY=ABS(A(IPT(3),N))/A(IPT(3),N)
      LL=(L-1)*2
404   W(L)=1.0/SQRT(A(IPT(3)+LL,N)*2.0
     C /(A(IPT(3)+LL,K+1)+A(IPT(3)+LL,K+2)))
     C *EXP((-1)**(L/2)*(A(IPT(4)+LL+1,N)
     C -0.5*(A(IPT(4)+LL+1,K+1)+A(IPT(4)+LL+1,K+2)))*FRIGIT)
      DO 405 J=1,4
405   COM(I,J)=CEXP(CMPLX(0.0,A(IPT(4)+2*(J-1),N)))*W(J)
401   COM(I,5)=CMPLX(A(IPT(INT),N),A(IPT(INT)+1,N))
      CALL CDSMSL(COM,4,4)
      DO 402 I=0,3
      A(IPT(4+INT)+2*I,K)=REAL(COM(I+1,5))
402   A(IPT(4+INT)+2*I+1,K)=AIMAG(COM(I+1,5))
      IF(INT.LT.2) GOTO 400
      ICOUNT=III
      DO 500 I=0,1
      DO 500 J=1,ICOUNT-3
      DO 500 K=0,6,2
      DVAR=SQRT(A(IPT(5+I+IP)+K,J)**2+A(IPT(5+I+IP)+K+1,J)**2)
      FACT=ATAN2(A(IPT(5+I+IP)+K+1,J),A(IPT(5+I+IP)+K,J))*57.2957795
      IF(FACT.LT.0.0) FACT=FACT+360.0
      A(IPT(7+I+IP)+K,J)=DVAR
500   A(IPT(7+I+IP)+K+1,J)=FACT
      WRITE(6,652)
      WRITE(6,657)
657   FORMAT(T10,'***THE AMPLITUDE AND PHASE DERIVED FROM THE WKB APPROX
     CIMATION FOR THE FIRST SOLUTION***')
      WRITE(6,658)
      WRITE(6,655) ((A(IPT(13),I)+1.5*DELT*BACK,(A(IPT(7+IP)+J,I),J=0,7)
     C            ),I=1,ICOUNT-3)
      WRITE(6,652)
      WRITE(6,659)
659   FORMAT(T10,'***THE AMPLITUDE AND PHASE DERIVED FROM THE WKB APPROX
     CIMATION FOR THE SECOND SOLUTION***')
      WRITE(6,658)
      WRITE(6,655) ((A(IPT(13),I)+1.5*DELT*BACK,(A(IPT(8+IP)+J,I),J=0,7)
     C            ),I=1,ICOUNT-3)
      WRITE(6,652)
      WRITE(6,660)
```

```
660 FORMAT(T10,'***THIS CHECKS THE ACCURACY OF THE WKB APPROXIMATION F
   COR AMPLITUDE***')
    WRITE(6,658)
    DO 502 I=1,ICOUNT-3
    DO 501 J=0,7,2
    F=0.5
    IF(J.EQ.0.OR.J.EQ.3) F=0.75
    X=A(IPT(13),I)+1.5*DELT*BACK
    G(J+1)=A(IPT(7+IP)+J,I)*EXP(-X*F)
501 G(J+2)=A(IPT(8+IP)+J,I)*EXP(-X*F)
502 WRITE(6,655) X,G
    WRITE(6,652)
    WRITE(6,661)
661 FORMAT(T10,'***ELIMINATES THE DOWN-GOING MAGNETIC WAVE ***')
    WRITE(6,658)
    CVT=57.2957795
    DO 300 I=1,ICOUNT-3
    DO 301 J=0,3
    P=A(IPT(7)+J*2+1,I)/CVT
    Q=A(IPT(8)+J*2+1,I)/CVT
    SOL1(J+1)=A(IPT(7)+J*2,I)*CEXP(CMPLX(0.0,P))
301 SOL2(J+1)=A(IPT(8)+J*2,I)*CEXP(CMPLX(0.0,Q))
    CVAR=SOL1(1)/SOL2(1)
    DO 302 J=1,4
302 SOL1(J)=SOL1(J)-CVAR*SOL2(J)
    DO 303 J=1,4
    G(2*J-1)=CABS(SOL1(J))
    G(2*J)=ATAN2(AIMAG(SOL1(J)),REAL(SOL1(J)))*CVT
303 IF(G(2*J).LT.0.0) G(2*J)=G(2*J)+360.0
300 WRITE(6,655) A(IPT(13),I)+1.5*DELT*BACK,(G(J),J=1,8)
  8 GOTO 2
  3 STOP
    END
```

```
      SUBROUTINE FCT(X,Y,DERY)
      COMMON BOUS,BRUNT,WNX,WNY,THETA,XTOP,XBOT,BACK
      COMPLEX AL(5), BE(5)
      REAL DERY(1),Y(1)
      REAL A(5)
    2 DVAR=EXP(-BACK*X)/A(5)*WNX2
      AL(4)=CMPLX(0.0,A(4))*BACK
      AL(3)=CMPLX(A(3)+DVAR,0.0)
      AL(2)=CMPLX(-BOUS*DVAR,A(2))*BACK
      AL(1)=CMPLX(A(1)+DVAR*B,0.0)
      DO 3 I=1,4
    3 BE(I)=CMPLX(Y(2*I-1),Y(2*I))
      DO 4 I=1,6
    4 DERY(I)=Y(I+2)
      DERY(7)=0.0
      DERY(8)=0.0
      DO 5 I=1,4
      DERY(7)=DERY(7)-REAL(AL(I)*BE(I))
    5 DERY(8)=DERY(8)-AIMAG(AL(I)*BE(I))
      RETURN
      ENTRY FFCT(X,Y,DERY)
      S=SIN(THETA)
      C=COS(THETA)
      WNX2=WNX*WNX
      WNY2=WNY*WNY
      WNH2=WNX2+WNY2
      B=(BRUNT*BRUNT-1.0)*WNH2
      A(5)=S*S
      A(4)=2.0*S*C*WNX
      A(3)=-WNX2-WNY2*S*S
      A(2)=-2.0*S*C*WNX*WNH2
      A(1)=WNX2*WNH2*C*C
      DO 1 I=1,4
    1 A(I)=A(I)/A(5)
      RETURN
      END
```

```
      SUBROUTINE OUT(X,Y,DERY,IHLF,NDIM,PRMT)
      COMMON BOUS,BRUNT,WNX,WNY,THETA,XTOP,XBOT,BACK
      COMMON A(104,90),IPT(13),ICOUNT,DELT,NMAX,IFST
      REAL Y(1),DERY(1),PRMT(1)
      COMPLEX PRES,WU,DWU,DDWU,DDDWU,IM,ACC,MAG,CI
    1 I=I+1
      IF(ABS((ICOUNT-1)*DELT-X+XBOT).GT.1.0E-5) RETURN
      DVAR=AMAX1(ABS(Y(1)),ABS(Y(2)),ABS(Y(3)),ABS(Y(4)))
      W=SQRT((Y(1)/DVAR)**2+(Y(2)/DVAR)**2)*DVAR
      DW=SQRT((Y(3)/DVAR)**2+(Y(4)/DVAR)**2)*DVAR
      WU=CMPLX(Y(1),Y(2))
      DWU=CMPLX(Y(3),Y(4))*BACK
      DDWU=CMPLX(Y(5),Y(6))
      DDDWU=CMPLX(Y(7),Y(8))*BACK
      PRES=IM*EXP(-BACK*BOUS*X)/WNH2
      ACC=CMPLX(S*EXP(X*BACK)/WNX2,0.0)
      ACC=ACC*(CI*(DDWU-WNH2*WU)+S*(DDDWU-WNH2*DWU))
      ACC=ACC+DWU
      PRES=PRES*ACC
      PRES=PRES*CONJG(WU)
      MAG=IM*EXP(-BACK*BOUS*X)/WNX2*EXP(BACK*X)
      ACC=(CI*WU+S*DWU)/WNH2
      ACC=ACC*CONJG(CI*DWU+S*DDWU)
      MAG=MAG*ACC
      F=REAL(PRES+MAG)
      PRMT(4)=0.0001*W
      IF(W.GT.1.0E20) PRMT(5)=1.0
      XX=X*BACK
      DO 2 J=0,7
    2 A(IPT(IFST)+J,ICOUNT)=Y(J+1)*BACK**(J/2)
      A(IPT(13),ICOUNT)=XX
      A(IPT(13)+IFST,ICOUNT)=F
      ICOUNT=ICOUNT+1
      RETURN
      ENTRY       FOUT(X,Y,DERY,IHLF,NDIM,PRMT)
      I=0
      WNX2=WNX*WNX
      WNY2=WNY*WNY
      WNH2=WNX2+WNY2
      WNH4=WNH2*WNH2
      IM=(0.0,1.0)
      S=SIN(THETA)
      C=COS(THETA)
      CI=C*WNX*IM
      RETURN
      END
```

```
      SUBROUTINE SERIES(X,BRUNT,WNX,WNY,C,S,U,N,TOL,DER)
      COMMON BOUS
      COMPLEX DER(4),P,TERM,T1,T2,T3,T4,DEN,IM,U
      X=EXP(-X)
      B=BRUNT*BRUNT-1.0
      W=WNX*WNX
      WH=W+WNY*WNY
      IM=(0.0,1.0)
      DO 4 J=1,4
    4 DER(J)=(0.0,0.0)
      T1=(1.0,0.0)
      T2=T1*U/X
      T3=T2*(U-1)/X
      T4=T3*(U-2)/X
      TERM=T1
      DO 1 I=1,N
      DER(1)=DER(1)+T1
      DER(2)=DER(2)+T2
      DER(3)=DER(3)+T3
      DER(4)=DER(4)+T4
      P=U+I
      DEN=(P*S-IM*C*WNX)**2*(P*P-WH)
      P=P-1
      TERM=-TERM*W*(P*P+BOUS*P+WH*B)/DEN*X
      T1=TERM
      T2=T1*(U+I)/X
      T3=T2*(U+I-1)/X
      T4=T3*(U+I-2)/X
      DVAR=2.0
      IF(I.GT.4) DVAR=AMAX1(CABS(T1/DER(1)),CABS(T2/DER(2)),
     C                     CABS(T3/DER(3)),CABS(T4/DER(4)))
    1 IF(DVAR.LT.TOL) GOTO 2
    2 DER(2)=-DER(2)*X
      DER(3)=DER(3)*X*X-DER(2)
      DER(4)=-DER(4)*X**3-3*DER(3)-2*DER(2)
      X=-ALOG(X)
      TOL=DVAR
      N=I
      DO 3 J=1,4
    3 DER(J)=DER(J)*CEXP(-U*X)
      RETURN
      END
```

```
      SUBROUTINE SECOND(X,BRUNT,WNX,WNY,C,S,U,N,TOL,DER)
      COMMON BOUS
      COMPLEX DER(4),P,TERM,T1,T2,T3,T4,DEN,IM,U
      COMPLEX FST,FST1,SEC,SEC1
      X=EXP(-X)
      B=BRUNT*BRUNT-1.0
      W=WNX*WNX
      WH=W+WNY*WNY
      IM=(0.0,1.0)
      DO 4 J=1,4
    4 DER(J)=(0.0,0.0)
      T1=(1.0,0.0)
      T2=T1*U/X
      T3=T2*(U-1)/X
      T4=T3*(U-2)/X
      FST=T1
      SEC=(0.0,0.0)
      DO 1 I=1,N
      DER(1)=DER(1)+T1
      DER(2)=DER(2)+T2
      DER(3)=DER(3)+T3
      DER(4)=DER(4)+T4
      P=U+I
      DEN=(P*S-IM*C*WNX)**2*(P*P-WH)
      P=P-1
      FST1=-FST*W*(P*P+BOUS*P+WH*B)/DEN*X
      SEC1=-SEC*W*(P*P+BOUS*P+WH*B)
     C      -FST*W*P
     C      -FST1*S*S*(P+1)*((P+1)*(P+1)-WH+P*(P+1))
      SEC1=SEC1/DEN*X
      T1=FST1*ALOG(X)+SEC1
      T2=((U+I)*T1+FST1)/X
      T3=((U+I-1)*T2+(U+I)*FST1)/X
      T4=((U+I-2)*T3+(U+I-1)*FST1)/X
      FST=FST1
      SEC=SEC1
      DVAR=2.0
      IF(I.GT.4) DVAR=AMAX1(CABS(T1/DER(1)),CABS(T2/DER(2)),
     C                     CABS(T3/DER(3)),CABS(T4/DER(4)))
    1 IF(DVAR.LT.TOL) GOTO 2
    2 DER(2)=-DER(2)*X
      DER(3)=DER(3)*X*X-DER(2)
      DER(4)=-DER(4)*X**3-3*DER(3)-2*DER(2)
      X=-ALOG(X)
      TOL=DVAR
      N=I
      DO 3 J=1,4
    3 DER(J)=DER(J)*CEXP(-U*X)
      RETURN
      END
```

```
      SUBROUTINE QUAD(BOUS,BR,WNX,WNY,X,A,S,C)
C                                                                         C
C*************************************************************************C
C                                                                         C
C        EVALUATES THE DISPERSION RELATION FOR MAGNETO-GRAVITO WAVES      C
C        THE ATMOSPHERE IS EXPONENTIAL AND ISOTHERMAL                     C
C        THE LEVEL Z=0 IS DEFINED AS THE LEVEL WHERE THE ALFVEN           C
C        FREQUENCY MATCHES THE WAVE FREQUENCY                             C
C        THE MAGNETIC FIELD IS UNIFORM AND INCLINED                       C
C        AT AN ARBITRARY ANGLE OF THETA                                   C
C                                                                         C
C        REQUIRES THE IBM SSP SUBROUTINE POLRT                            C
C                                                                         C
C        WNX       THE MERIDIONAL WAVE NUMBER                             C
C        WNY       HORIZONTAL TRANSVERSE WAVE NUMBER                      C
C        THETA     THE MAGNETIC INCLINATION                               C
C        BRUNT     RATIO OF BRUNT TO WAVE FREQUENCY                       C
C        BOUS      .NE.0.0  MODELLED IN AN INCOMPRESSIBLE FLUID           C
C                                                                         C
C*************************************************************************C
C                                                                         C
      REAL PR(5),PI(5),ZEROR(5),ZEROI(5),A(10)
      COMPLEX POLY(5),ROOT,P,DER,TOL
      BRUNT=BR*BR-1.0
      IF(C.LT.1.0E-4) C=0.0
      Y=WNY*WNY/WNX/WNX
      H=Y+1
      DVAR=EXP(-X)
      DO 12 I=1,5
   12 PI(I)=0.0
      PR(1)=(DVAR*BRUNT+C*C)*H
      PR(2)=2.0*S*C*H
      IF(BOUS.NE.0.0) PI(2)=-DVAR/WNX
      PR(3)=S*S*Y+1.0-DVAR
      PR(4)=2.0*S*C
      PR(5)=S*S
      DO 1 I=1,5
    1 POLY(I)=CMPLX(PR(I),PI(I))/PR(5)
      CALL POLRT(PR,PI,4,ZEROR,ZEROI,IER)
      DO 2 I=1,5
      A(2*I-1)=ZEROR(I)
    2 A(2*I)=ZEROI(I)
      CALL SORT(A,8)
      IF(BOUS.EQ.0.0) GOTO 8
      WGT=1.0
      NDIM=5
      DO 7 I=1,4
      ROOT=CMPLX(A(2*I-1),A(2*I)+X/4.0/WNX)
      K=0
```

```
    5 K=K+1
      P=(0.0,0.0)
      DER=(0.0,0.0)
      DO 4 J=1,NDIM
      P=P+POLY(J)*ROOT**(J-1)
    4 DER=DER+(J-1)*POLY(J)*ROOT**(J-2)
      TOL=P/DER
      ROOT=ROOT-TOL*WGT
      IF(CABS(TOL/ROOT).GT.1.E-6.AND.K.LT.40) GOTO 5
      A(2*I-1)=REAL(ROOT)
      A(2*I)=AIMAG(ROOT)
      DO 10 L=1,NDIM-1
   10 POLY(NDIM-L)=POLY(NDIM-L)+ROOT*POLY(NDIM-L+1)
      NDIM=NDIM-1
      DO 11 L=1,NDIM
   11 POLY(L)=POLY(L+1)
      POLY(NDIM+1)=(0.0,0.0)
    7 CONTINUE
      CALL SORT(A,8)
    8 DO 6 I=1,10
      A(I)=A(I)*WNX
    6 IF(ABS(A(I)).LT.1.0E-6) A(I)=0.0
      RETURN
      END
```

```
      SUBROUTINE SORT(A,N)
      REAL A(1)
      DO 2 M=1,N,2
      D=1.0E60
      DO 1 I=M,N,2
      IF(A(I).LT.D) IMIN=I
    1 IF(A(I).LT.D) D=A(I)
      X=A(IMIN)
      Y=A(IMIN+1)
      A(IMIN)=A(M)
      A(IMIN+1)=A(M+1)
      A(M)=X
    2 A(M+1)=Y
      RETURN
      END
```

SE

```
CC      TRANSLATE TO BCD

        SUBROUTINE CDSMSL (A,KK,LL)
C
C       SUBROUTINE CDSMSL COMPLEX VERSION
C
C       REFERENCES
C          ''SUBROUTINES TO SOLVE AN INDEPENDENT SET OF LINEAR
C            SIMULTANEOUS EQUATIONS'' HS/FEB/PAW-84, 21 JULY 1965.
C
C       PURPOSE
C       TO SOLVE A SET OF SIMULTANEOUS LINEAR COMPLEX EQUATIONS, AX=B.
C
C       USAGE
C       CALL CDSMSL(A,N,ND1)
C
C       DESCRIPTION OF PARAMETERS
C       A-IS A 2-DIMENSIONAL(ND1*ND2) MATRIX OF COEFFICIENTS
C       N- IS THE NUMBER OF EQUATIONS AND UNKNOWNS.
C       ND2- IS THE FIRST DIMENSION OF A IN THE CALLING PROGRAM.
C       (ND1.GE.N AND ND2.GE.N+1)
C
C       CALLING PROGRAM SETUP
C       A(I,J) FOR I,J=1,N
C       A(I,N+1)=B(I) FOR I=1,N
C       THE SOLUTION IS RETURNED IN COLUMN N+1 OF MATRIX A
C       MATRIX A IS DESTROYED BY THE SUBROUTINE.
C
C       REMARKS
C       IF MATRIX IS SINGULAR, AN ERROR MESSAGE IS PRINTED
C       AND THE JOB IS TERMINATED.
C
C       SUBROUTINES REQUIRED
C       NONE.
C
C       METHOD
C       SOLUTION IS OBTAINED BY ELIMINATION USING LARGEST PIVOTAL
C       DIVISOR IN THE COLUMN. EACH STAGE OF ELIMINATION CONSISTS
C       OF INTERCHANGING  ROWS WHEN NECESSARY TO AVOID DIVISION BY
C       ZERO OR SMALL NUMBERS.
C       THE FORWARD SOLUTION TO OBTAIN VARIABLE N IS DONE IN N
C       STAGES. THE BACK SOLUTION FOR THE OTHER VARIABLES IS
C       CALCULATED BY SUCCESSIVE SUBSTITUTIONS. FINAL  SOLUTION
C       VALUES ARE DEVELOPED IN COLUMN N+1 OF MATRIX A, WITH
C       VARIABLES 1 IN A(1,N+1), VARIABLE 2 IN A(2,N+1),........,
C       VARIABLE N IN A(N,N+1).
C
C       ******************************************************************
```

```
C      *********************************************************************
       COMPLEX     A(LL,1),B,BIG,ZERO
       ZERO = (0.0,0.0)
       N  = KK
       N1 = N+1
       DO 50  L=1,N
       L1 = L+1
       BIG = ZERO
       DO 25  I=L,N
       IF (CABS(A(I,L)).LE.CABS(BIG)) GO TO 25
       K = I
       BIG = A(I,L)
    25 CONTINUE
       IF (CABS(BIG).NE.0.00) GO TO 30
       WRITE(6,26)
    26  FORMAT(24H CDSMSL MATRIX SINGULAR.)
       STOP
    30 DO 40  J=L,N1
       IF (K.EQ.L) GO TO 40
       B       = A(K,J)
       A(K,J)  = A(L,J)
       A(L,J)  = B
    40 A(L,J)  = A(L,J)/BIG
       IF (L.EQ.N) GO TO 50
       DO 48  I=L1,N
       IF (CABS(A(I,L)).EQ.0.00) GO TO 48
       DO 45  J=L1,N1
    45 A(I,J)  = A(I,J)-A(I,L)*A(L,J)
    48 CONTINUE
    50 CONTINUE
       IF (N.EQ.1) RETURN
       N2 = N-1
       DO 60  L=1,N2
       I  = N-L
       L1 = I+1
       DO 60  J=L1,N
    60 A(I,N1)  = A(I,N1)-A(I,J)*A(J,N1)
       RETURN
       END
```

MODELLED FOR AN INCOMPRESSIBLE FLUID

*** SAMPLE PROGRAMME ***

THE RATIO OF BRUNT FREQUENCY TO WAVE FREQUENCY 1.200
THE RATIO OF MERIDIONAL SCALE TO SCALE HEIGHT 4.250
RATIO OF TRANSVERSE SCALE TO SCALE HEIGHT 0.
THE INCLINATION OF THE MAGNETIC FIELD -60.00
THE INTEGRATION COMMENCES AT 0. AND TERMINATES AT -4.000

THE INITIAL SOLUTION
F(X) AND ITS DERIVATIVES FOR U=I*KX*COT(THETA)
THE FINAL COLUMN TABULATES THE FLUX

0.00	-0.450625	-0.682131	2.12926	-2.66556	14.8920	7.47746	-34.4838	82.1209	-2.165120E-08
-0.10	-0.585351	-0.392559	0.548904	-2.97771	15.8576	-1.36287	16.6843	89.6515	-2.095317E-08
-0.20	-0.565958	-0.115910	-0.865147	-2.42144	11.5802	-9.33581	66.8814	64.0955	-2.046022E-08
-0.30	-0.434218	7.087541E-02	-1.63021	-1.24969	3.27255	-13.1728	93.4566	8.85564	-2.020261E-08
-0.40	-0.270345	0.131249	-1.49672	2.081038E-03	-5.68059	-10.7736	77.7041	-55.9028	-2.017032E-08
-0.50	-0.160020	8.869971E-02	-0.623805	0.714501	-10.7794	-2.77871	18.2418	-97.1100	-2.031788E-08
-0.60	-0.151337	1.978961E-02	0.415349	0.502059	-8.69594	6.81340	-59.3802	-84.1406	-2.056413E-08
-0.70	-0.223789	1.530168E-02	0.885368	-0.500503	0.110864	12.0234	-107.491	-11.9086	-2.081651E-08
-0.80	-0.293847	0.123453	0.343438	-1.60210	10.3086	8.41368	-82.2748	82.5379	-2.100501E-08
-0.90	-0.266364	0.308998	-0.956237	-1.91772	14.0063	-2.91481	17.3213	130.332	-2.110824E-08
-1.00	-0.108584	0.465304	-2.07249	-1.01695	6.46051	-14.2210	126.985	77.8888	-2.113757E-08
-1.10	0.107347	0.487828	-1.99554	0.593418	-8.35373	-15.7792	147.283	-52.5155	-2.124102E-08
-1.20	0.243848	0.363576	-0.565506	1.70623	-18.2891	-4.83029	31.2767	-149.087	-2.165893E-08
-1.30	0.210999	0.193632	1.12940	1.45918	-12.7572	8.88347	-136.519	-96.1058	-2.278307E-08
-1.40	6.132109E-02	0.101476	1.56603	0.364259	4.79274	9.98443	-180.957	81.9423	-2.454885E-08
-1.50	-4.552540E-02	9.477259E-02	0.372562	2.349424E-02	16.4214	-4.89850	-24.8097	182.111	-2.614055E-08
-1.60	-5.518768E-03	4.066340E-02	-1.04025	1.27730	8.68086	-17.5826	159.149	28.8306	-2.760538E-08
-1.70	0.113736	-0.168112	-1.07169	2.71274	-7.20777	-6.58095	109.153	-236.260	-3.079344E-08
-1.80	0.175903	-0.427998	-0.181109	2.02607	-6.63854	20.2435	-120.442	-230.100	-3.508876E-08
-1.90	0.185945	-0.503023	-0.289580	-0.647850	8.85127	27.3101	-120.254	115.283	-3.818013E-08
-2.00	0.267535	-0.333937	-1.28996	-2.31348	5.52347	2.86952	205.300	296.245	-4.459275E-08
-2.10	0.380493	-0.130079	-0.484517	-1.45671	-22.4853	-15.0106	249.174	8.64386	-5.279503E-08
-2.20	0.292370	-4.737563E-02	2.27709	-0.419675	-23.8432	-2.52663	-265.464	-174.384	-5.630568E-08
-2.30	6.813338E-03	2.424296E-03	2.71788	-0.663043	17.5131	2.32411	-403.064	111.348	-6.645507E-08
-2.40	-0.132971	5.464199E-02	-8.846960E-02	-0.111725	26.9660	-12.5633	264.643	60.9126	-6.692164E-08
-2.50	-5.245598E-02	1.267317E-02	-0.798569	0.657930	-14.2648	4.44755	348.326	-343.332	-7.749495E-08
-2.60	-7.355773E-02	1.876516E-02	1.21099	-1.09802	-13.9702	21.7814	-326.395	166.855	-7.805147E-08
-2.70	-0.205561	0.179245	0.906230	-1.32376	15.4516	-22.8107	-73.2402	463.469	-7.854237E-08
-2.80	-0.230316	0.159075	-0.134306	1.64543	0.768049	-17.5524	181.532	-600.281	-9.477890E-08
-2.90	-0.223910	1.640208E-02	-8.165963E-02	0.247174	5.17575	35.7340	-186.073	-44.1104	-1.248407E-07
-3.00	-0.174989	0.118806	-0.808395	-1.34271	-0.586067	-16.9295	341.653	663.073	-1.246901E-07
-3.10	-0.147043	0.112077	0.500056	1.41157	-12.7847	-13.5170	-364.316	-678.747	-1.264223E-07
-3.20	-0.175856	7.366663E-03	-0.748603	-3.701529E-04	30.8952	23.8176	43.6730	279.825	-1.297894E-07
-3.30	-2.106457E-02	5.063942E-02	-1.17805	-0.100385	-31.9336	-16.8143	512.242	35.3497	-1.317660E-07
-3.40	-5.347933E-02	8.799334E-03	0.865104	0.456646	19.6988	7.24569	-963.899	-71.0738	-1.356293E-07
-3.50	2.459390E-02	-1.657312E-02	-1.97639	0.293393	-4.66415	-3.07810	1191.34	-109.229	-1.392209E-07
-3.60	7.617358E-02	-2.672112E-02	1.02042	-0.336152	-8.18733	6.67068	-1206.68	365.198	-1.394476E-07
-3.70	7.809100E-02	-3.197534E-02	-1.41289	0.849279	9.88343	-10.7287	1134.69	-654.580	-1.418239E-07
-3.80	0.129321	-5.531273E-02	0.743747	-0.945654	-8.76487	13.7177	-1008.33	1000.50	-1.421282E-07
-3.90	0.120536	-4.368868E-02	-0.686287	1.21918	-0.798870	-7.29722	778.272	-1400.52	-1.435498E-07
-4.00	0.126948	-4.017723E-02	0.366956	-1.31887	4.51372	-3.73032	-352.497	1679.03	-1.441590E-07

THE INITIAL SOLUTION
F(X) AND ITS DERIVATIVES FOR U=KH
THE FINAL COLUMN TABULATES THE FLUX

0.00	-7.550922E-02	0.321796	0.814902	-0.980415	-4.92818	1.75638	22.4204	5.50877
-0.10	-0.185677	0.427309	1.43179	-1.11227	-7.52771	0.691146	29.4658	16.9305
-0.20	-0.371653	0.538471	2.34150	-1.06801	-10.7464	-1.90226	34.1743	36.5237
-0.30	-0.665245	0.628559	3.58672	-0.649653	-14.1108	-6.96679	31.2011	66.6497
-0.40	-1.09912	0.645999	5.13010	0.444428	-16.4303	-15.5984	11.3895	107.584
-0.50	-1.69456	0.503709	6.76231	2.61941	-15.3847	-28.6801	-38.6386	154.054
-0.60	-2.43794	7.092617E-02	7.96972	6.32677	-7.16770	-46.0724	-134.598	189.832
-0.70	-3.24267	-0.824205	7.78278	11.8964	13.4601	-65.1926	-287.708	181.239
-0.80	-3.89773	-2.36713	4.67620	19.1962	52.0516	-79.0228	-489.699	73.1356
-0.90	-4.01466	-4.68500	-3.32746	27.0857	111.362	-74.1572	-688.057	-204.474
-1.00	-3.00495	-7.71231	-18.1076	32.7407	185.448	-30.5815	-756.853	-705.840
-1.10	-0.146497	-10.9944	-40.1737	31.1680	251.399	73.6774	-483.842	-1400.90
-1.20	5.18108	-13.4791	-66.5792	15.5954	262.311	249.070	382.813	-2070.15
-1.30	13.0293	-13.4320	-88.5385	-20.2500	150.350	470.400	1969.92	-2210.89
-1.40	22.2244	-8.71334	-90.4197	-77.1439	-146.379	648.879	3971.37	-1083.42
-1.50	29.7996	2.32595	-53.1016	-143.281	-622.856	623.979	5311.75	1905.56
-1.60	31.1157	19.2811	35.2565	-189.048	-1125.65	216.298	4160.89	6370.77
-1.70	21.4322	38.0237	161.700	-171.712	-1315.37	-624.765	-1101.83	10001.2
-1.80	-0.796281	50.3731	274.132	-58.5332	-788.350	-1617.39	-9707.43	8651.56
-1.90	-30.1872	46.9897	290.963	134.050	564.825	-2074.24	-16335.9	-907.961
-2.00	-53.7229	23.9622	153.368	312.824	2123.88	-1252.77	-12461.1	-15594.6
-2.10	-56.9587	-10.4969	-96.3986	341.679	2569.29	801.714	5464.15	-22884.9
-2.20	-36.3106	-37.1172	-289.360	160.111	954.024	2598.66	25142.1	-9095.56
-2.30	-7.06502	-39.7744	-250.888	-98.3395	-1669.09	2077.85	21693.7	19808.4
-2.40	7.18841	-23.7261	-22.6466	-174.970	-2330.30	-715.069	-11485.2	29451.7
-2.50	1.12947	-13.7931	99.0480	3.34967	286.224	-2275.72	-33887.6	-4299.72
-2.60	-2.36421	-22.9190	-67.0808	136.876	2466.84	190.107	-520.031	-37895.4
-2.70	14.4709	-29.8207	-226.709	-45.5356	-30.1240	2865.41	42738.1	-3227.34
-2.80	30.3199	-13.1311	-36.7118	-241.503	-3085.41	156.818	2884.91	47828.2
-2.90	21.2519	4.53427	163.397	-56.6108	56.6060	-2970.57	-52051.4	-3230.11
-3.00	12.4752	-0.611470	-34.8555	91.1068	2687.92	920.859	17662.5	-53616.4
-3.10	22.9864	1.15664	-89.1318	-151.198	-2161.45	2266.35	47916.0	39940.3
-3.20	17.5001	18.4133	184.407	-96.4421	-1303.22	-3153.97	-63626.9	26870.6
-3.30	3.23222	13.6705	18.1997	-124.262	3224.28	579.659	15105.6	-73366.6
-3.40	9.54101	12.4404	-33.5698	-123.359	-2816.16	1810.75	47135.0	66614.4
-3.50	-1.42175	21.3708	182.720	45.3114	1010.55	-3550.57	-86827.6	-23349.1
-3.60	-6.10397	9.57249	-91.6096	59.0675	1071.51	3416.89	93471.2	-27732.9
-3.70	-4.98821	15.4816	134.297	-53.9998	-2231.04	-3029.06	-83003.1	66482.1
-3.80	-14.0613	6.07901	-59.8483	137.741	3161.00	2363.39	70796.2	-92146.1
-3.90	-8.56906	7.14453	78.6309	-68.0237	-3189.46	-2253.82	-70079.9	104612.
-4.00	-15.1140	-0.958353	-81.5020	130.250	3233.77	2691.49	88200.6	-105613.

REAL AND IMAGINARY PARTS OF THE VERTICAL WAVE NUMBER

0.00	-2.21024	3.71173	-1.69782	-3.71729	2.72412	-0.551970	6.09142	0.557531
-0.10	-2.36677	3.68363	-1.82203	-3.68538	2.73014	-0.547948	6.36613	0.549693
-0.20	-2.53147	3.65190	-1.95194	-3.64999	2.73540	-0.544081	6.65549	0.542167
-0.30	-2.70481	3.61574	-2.08759	-3.61036	2.74000	-0.540403	6.95987	0.535021
-0.40	-2.88730	3.57421	-2.22893	-3.56555	2.74401	-0.536954	7.27971	0.528298
-0.50	-3.07958	3.52620	-2.37593	-3.51449	2.74751	[illegible]	[illegible]	[illegible]
-0.60	-3.28238	3.47043	-2.52845	-3.45590	2.75058	-0.530742	7.96773	0.516214
-0.70	-3.49658	3.40539	-2.68627	-3.38827	2.75326	-0.527982	8.33707	0.510863
-0.80	-3.72330	3.32927	-2.84902	-3.30979	2.75562	-0.525444	8.72417	0.505965
-0.90	-3.96391	3.23990	-3.01608	-3.21829	2.75769	-0.523116	9.12978	0.501507
-1.00	-4.22026	3.13465	-3.18647	-3.11113	2.75951	-0.520988	9.55470	0.497469
-1.10	-4.49483	3.01030	-3.35860	-2.98509	2.76112	-0.519044	9.99979	0.493833
-1.20	-4.79120	2.86289	-3.52985	-2.83619	2.76254	-0.517273	10.4660	0.490575
-1.30	-5.11476	2.68765	-3.69586	-2.65966	2.76380	-0.515660	10.9543	0.487673
-1.40	-5.47408	2.47913	-3.84915	-2.45005	2.76491	-0.514194	11.4658	0.485104
-1.50	-5.88340	2.23261	-3.97664	-2.20259	2.76590	-0.512861	12.0016	0.482846
-1.60	-6.36564	1.94920	-4.05662	-1.91842	2.76678	-0.511651	12.5630	0.480875
-1.70	-6.94939	1.64885	-4.06182	-1.61747	2.76757	-0.510553	13.1511	0.479171
-1.80	-7.64314	1.37762	-3.98510	-1.34577	2.76827	-0.509557	13.7675	0.477712
-1.90	-8.41371	1.17187	-3.86111	-1.13970	2.76889	-0.508654	14.4134	0.476480
-2.00	-9.22214	1.02837	-3.73033	-0.995991	2.76945	-0.507836	15.0905	0.475455
-2.10	-10.0517	0.928064	-3.61112	-0.895590	2.76995	-0.507094	15.8003	0.474619
-2.20	-10.9000	0.855535	-3.50748	-0.823070	2.77040	-0.506422	16.5446	0.473958
-2.30	-11.7699	0.801121	-3.41850	-0.768761	2.77080	-0.505813	17.3251	0.473453
-2.40	-12.6652	0.758968	-3.34211	-0.726798	2.77116	-0.505262	18.1437	0.473092
-2.50	-13.5900	0.725441	-3.27631	-0.693539	2.77148	-0.504763	19.0023	0.472861
-2.60	-14.5481	0.698195	-3.21935	-0.666630	2.77177	-0.504311	19.9032	0.472746
-2.70	-15.5431	0.675660	-3.16980	-0.644494	2.77203	-0.503901	20.8483	0.472736
-2.80	-16.5785	0.656747	-3.12652	-0.626036	2.77227	-0.503531	21.8402	0.472820
-2.90	-17.6576	0.640680	-3.08855	-0.610472	2.77248	-0.503196	22.8812	0.472987
-3.00	-18.7839	0.626886	-3.05512	-0.597224	2.77267	-0.502892	23.9738	0.473230
-3.10	-19.9605	0.614940	-3.02560	-0.585860	2.77284	-0.502617	25.1207	0.473537
-3.20	-21.1909	0.604513	-2.99946	-0.576047	2.77299	-0.502368	26.3249	0.473902
-3.30	-22.4786	0.595350	-2.97626	-0.567525	2.77313	-0.502143	27.5892	0.474318
-3.40	-23.8269	0.587251	-2.95562	-0.560082	2.77326	-0.501939	28.9167	0.474776
-3.50	-25.2395	0.580055	-2.93723	-0.553571	2.77337	-0.501755	30.3108	0.475272
-3.60	-26.7202	0.573630	-2.92081	-0.547840	2.77348	-0.501588	31.7750	0.475798
-3.70	-28.2727	0.567869	-2.90614	-0.542783	2.77357	-0.501437	33.3128	0.476351
-3.80	-29.9012	0.562684	-2.89300	-0.538309	2.77365	-0.501300	34.9280	0.476925
-3.90	-31.6098	0.558002	-2.88124	-0.534341	2.77373	-0.501177	36.6247	0.477515
-4.00	-33.4028	0.553760	-2.87068	-0.530814	2.77380	-0.501065	38.4072	0.478119

REAL AND IMAGINARY PARTS OF THE PHASE INTEGRAL

0.00	0.	0.	0.	0.	0.	0.	0.	0.
-0.10	0.228785	-0.369795	0.175945	0.370159	-0.272720	5.499486E-02	-0.622758	-5.535899E-02
-0.20	0.473627	-0.736605	0.364597	0.736959	-0.546003	0.109595	-1.27372	-0.109949
-0.30	0.735368	-1.10003	0.566526	1.10002	-0.819778	0.163818	-1.95436	-0.163805
-0.40	1.01490	-1.45957	0.782305	1.45886	-1.09398	0.217684	-2.66621	-0.216967
-0.50	1.31316	-1.81465	1.01250	1.81292	-1.36856	0.271216	-3.41083	-0.269480
-0.60	1.63116	-2.16455	1.25768	2.16150	-1.64347	0.324438	-4.18986	-0.321388
-0.70	1.97001	-2.50843	1.51837	2.50379	-1.91867	0.377372	-5.00495	-0.372738
-0.80	2.33090	-2.84526	1.79509	2.83880	-2.19411	0.430042	-5.85786	-0.423576
-0.90	2.71514	-3.17384	2.08832	3.16532	-2.46978	0.482468	-6.75041	-0.473946
-1.00	3.12421	-3.49271	2.39842	3.48193	-2.74564	0.534672	-7.68447	-0.523891
-1.10	3.55980	-3.80013	2.72567	3.78691	-3.02167	0.586672	-8.66202	-0.573453
-1.20	4.02390	-4.09400	3.07011	4.07818	-3.29786	0.638486	-9.68513	-0.622670
-1.30	4.51895	-4.37178	3.43147	4.35323	-3.57418	0.690132	-10.7560	-0.671580
-1.40	5.04805	-4.63041	3.80887	4.60901	-3.85061	0.741623	-11.8768	-0.720216
-1.50	5.61543	-4.86632	4.20044	4.84196	-4.12715	0.792975	-13.0499	-0.768611
-1.60	6.22716	-5.07568	4.60261	5.04827	-4.40379	0.844199	-14.2779	-0.816795
-1.70	6.89198	-5.25556	5.00924	5.22505	-4.68051	0.895309	-15.5634	-0.864795
-1.80	7.62078	-5.40643	5.41217	5.37276	-4.95730	0.946313	-16.9091	-0.912637
-1.90	8.42318	-5.53333	5.80468	5.49645	-5.23416	0.997223	-18.3179	-0.960345
-2.00	9.30476	-5.64291	6.18420	5.60280	-5.51108	1.04805	-19.7928	-1.00794
-2.10	10.2683	-5.74045	6.55115	5.69710	-5.78805	1.09879	-21.3371	-1.05544
-2.20	11.3157	-5.82945	6.90695	5.78285	-6.06506	1.14947	-22.9541	-1.10287
-2.30	12.4490	-5.91216	7.25313	5.86232	-6.34212	1.20008	-24.6472	-1.15024
-2.40	13.6706	-5.99008	7.59107	5.93701	-6.61922	1.25063	-26.4204	-1.19757
-2.50	14.9831	-6.06424	7.92191	6.00797	-6.89635	1.30113	-28.2773	-1.24486
-2.60	16.3897	-6.13538	8.24663	6.07593	-7.17352	1.35159	-30.2222	-1.29214
-2.70	17.8939	-6.20403	8.56603	6.14145	-7.45071	1.40200	-32.2594	-1.33941
-2.80	19.4996	-6.27063	8.88080	6.20495	-7.72792	1.45237	-34.3935	-1.38669
-2.90	21.2111	-6.33548	9.19151	6.26676	-8.00516	1.50270	-36.6291	-1.43398
-3.00	23.0328	-6.39884	9.49866	6.32712	-8.28242	1.55301	-38.9714	-1.48129
-3.10	24.9695	-6.46092	9.80266	6.38626	-8.55969	1.60328	-41.4257	-1.52863
-3.20	27.0267	-6.52188	10.1039	6.44435	-8.83698	1.65353	-43.9975	-1.57600
-3.30	29.2096	-6.58186	10.4027	6.50152	-9.11429	1.70376	-46.6927	-1.62341
-3.40	31.5244	-6.64099	10.6992	6.55789	-9.39161	1.75396	-49.5174	-1.67087
-3.50	33.9772	-6.69934	10.9939	6.61356	-9.66894	1.80415	-52.4783	-1.71837
-3.60	36.5746	-6.75702	11.2867	6.66863	-9.94628	1.85431	-55.5820	-1.76592
-3.70	39.3236	-6.81409	11.5781	6.72316	-10.2236	1.90447	-58.8357	-1.81353
-3.80	42.2317	-6.87062	11.8680	6.77721	-10.5010	1.95460	-62.2471	-1.86119
-3.90	45.3065	-6.92665	12.1567	6.83083	-10.7784	2.00473	-65.8240	-1.90891
-4.00	48.5564	-6.98223	12.4443	6.88409	-11.0557	2.05484	-69.5749	-1.95670

THE AMPLITUDE AND PHASE DERIVED FROM THE WKB APPROXIMATION FOR THE FIRST SOLUTION

-0.15	2.804182E-02	81.3343	3.002792E-02	255.886	0.479451	254.501	0.314511	210.082
-0.25	4.910241E-02	87.5219	5.412912E-02	266.446	0.448602	253.566	0.287704	210.224
-0.35	7.249221E-02	82.0183	8.173881E-02	264.082	0.421491	252.025	0.262600	209.716
-0.45	9.146148E-02	76.5455	0.104591	261.571	0.401757	250.204	0.242779	208.564
-0.55	0.106473	76.0810	0.122412	264.210	0.391386	248.947	0.229504	207.887
-0.65	0.127042	80.7809	0.145894	272.646	0.388103	249.336	0.217590	208.975
-0.75	0.161993	85.2719	0.186786	281.582	0.384826	252.032	0.199245	211.185
-0.85	0.205794	86.4324	0.240127	287.301	0.371901	256.802	0.174963	211.407
-0.95	0.246262	86.4655	0.290970	291.799	0.340890	262.368	0.158754	207.662
-1.05	0.280417	88.1864	0.333925	298.125	0.289889	266.055	0.158232	205.700
-1.15	0.315616	91.6707	0.377524	306.739	0.232499	262.955	0.153843	210.028
-1.25	0.350900	94.8489	0.423172	315.540	0.205525	249.908	0.131533	213.618
-1.35	0.369730	97.3757	0.451108	323.677	0.229029	239.909	0.114635	208.039
-1.45	0.361844	100.887	0.444080	332.469	0.259115	242.290	0.117849	206.237
-1.55	0.333562	105.606	0.407298	342.114	0.255239	249.920	0.108931	211.077
-1.65	0.291430	109.507	0.354389	349.857	0.225780	253.826	9.627593E-02	209.233
-1.75	0.241292	112.072	0.295970	353.338	0.207477	251.921	9.223803E-02	209.128
-1.85	0.196463	114.340	0.246263	353.168	0.205484	251.078	8.379956E-02	209.406
-1.95	0.162376	115.809	0.215620	350.874	0.198506	252.986	7.936933E-02	208.540
-2.05	0.135851	117.007	0.198904	347.758	0.182945	254.439	7.354943E-02	209.855
-2.15	0.117188	118.452	0.186274	345.449	0.168309	254.330	6.754219E-02	209.153
-2.25	0.102907	119.016	0.176080	344.909	0.159586	253.587	6.342940E-02	209.248
-2.35	9.111123E-02	119.751	0.168170	344.923	0.153943	253.151	5.854341E-02	209.384
-2.45	8.157111E-02	119.939	0.158796	344.984	0.146956	253.235	5.463651E-02	209.244
-2.55	7.311607E-02	120.431	0.149068	345.654	0.138440	253.575	5.058485E-02	209.403
-2.65	6.609819E-02	120.543	0.140491	345.897	0.131105	253.586	4.717317E-02	209.252
-2.75	5.995095E-02	120.930	0.131500	345.718	0.125366	253.206	4.370669E-02	209.391
-2.85	5.441268E-02	120.922	0.124215	345.707	0.119293	253.190	4.073892E-02	209.358
-2.95	4.968179E-02	121.057	0.118010	345.465	0.113016	253.419	3.779677E-02	209.277
-3.05	4.528010E-02	121.315	0.111691	345.748	0.107774	253.208	3.511780E-02	209.463
-3.15	4.137522E-02	121.309	0.105786	345.830	0.102625	253.191	3.274016E-02	209.389
-3.25	3.796250E-02	121.299	9.994484E-02	345.747	9.741310E-02	253.353	3.038653E-02	209.296
-3.35	3.484402E-02	121.475	9.508969E-02	345.743	9.283147E-02	253.220	2.817947E-02	209.342
-3.45	3.198379E-02	121.692	9.005201E-02	345.854	8.818457E-02	253.343	2.618500E-02	209.566
-3.55	2.934527E-02	121.831	8.545481E-02	345.791	8.389774E-02	253.288	2.439132E-02	209.723
-3.65	2.688884E-02	122.253	8.117406E-02	345.896	7.971069E-02	253.377	2.274947E-02	210.203
-3.75	2.645914E-02	119.620	7.727221E-02	346.230	7.535014E-02	253.446	2.184924E-02	204.909
-3.85	2.299306E-02	121.154	7.305499E-02	345.869	7.214812E-02	253.274	1.960503E-02	208.729

THE AMPLITUDE AND PHASE DERIVED FROM THE WKB APPROXIMATION FOR THE SECOND SOLUTION

-0.15	7.07771	269.177	7.85807	84.2190	1.92961	232.800	0.547037	96.7410
-0.25	9.20652	270.527	10.2939	88.6195	2.46619	263.962	0.654807	150.660
-0.35	11.8943	271.934	13.3930	93.2597	3.12967	296.102	0.778781	207.246
-0.45	15.2434	273.401	17.2856	98.1529	3.93787	329.267	0.919263	266.636
-0.55	19.3505	274.934	22.1001	103.316	4.90394	3.50932	1.07534	328.978
-0.65	24.2901	276.540	27.9443	108.768	6.03084	38.8855	1.24419	34.4291
-0.75	30.0873	278.231	34.8742	114.537	7.30325	75.4607	1.42015	103.166
-0.85	36.6808	280.025	42.8496	120.656	8.67601	113.308	1.59355	175.386
-0.95	43.8720	281.950	51.6702	127.169	10.0590	152.509	1.74934	251.316
-1.05	51.2654	284.051	60.8962	134.133	11.2986	193.156	1.86562	331.232
-1.15	58.2074	286.398	69.7576	141.617	12.1595	235.352	1.91245	55.4997
-1.25	63.7510	289.094	77.0785	149.702	12.3135	279.194	1.85214	144.652
-1.35	66.7053	292.281	81.2858	158.443	11.3641	324.719	1.64510	239.604
-1.45	65.9046	296.077	80.6982	167.741	8.98227	11.6899	1.27370	342.172
-1.55	60.9061	300.317	74.5186	176.942	5.31901	58.8332	0.805912	96.0095
-1.65	52.8204	304.231	64.3347	184.408	1.51354	97.9819	0.406448	224.776
-1.75	43.8635	307.139	53.5595	188.356	1.18897	356.248	0.149736	29.0950
-1.85	35.8372	309.104	44.9447	188.284	2.07078	34.8918	0.201677	214.092
-1.95	29.4444	310.596	39.3577	185.506	1.75085	82.0352	0.211221	342.012
-2.05	24.7354	311.998	36.0294	182.481	1.10167	133.896	0.147403	103.683
-2.15	21.3117	313.173	33.8569	180.524	0.584406	192.236	3.706490E-02	243.783
-2.25	18.6996	313.988	32.1298	179.693	0.271657	264.018	4.863292E-02	31.9204
-2.35	16.5881	314.508	30.5064	179.644	0.150101	2.72898	2.962825E-02	196.665
-2.45	14.8164	314.878	28.8572	180.003	0.150025	97.1768	2.317751E-02	11.2076
-2.55	13.3079	315.203	27.1673	180.447	0.150493	171.412	2.090704E-02	185.554
-2.65	12.0165	315.490	25.4927	180.717	0.122229	241.268	1.808037E-02	3.48537
-2.75	10.8972	315.698	23.9385	180.682	7.227656E-02	316.470	1.382103E-02	194.418
-2.85	9.90748	315.826	22.5967	180.465	2.334291E-02	72.5216	1.003172E-02	51.8737
-2.95	9.02243	315.942	21.4424	180.388	3.901570E-02	236.114	1.035270E-02	291.614
-3.05	8.23522	316.083	20.3361	180.566	5.143511E-02	327.782	1.234258E-02	166.520
-3.15	7.53523	316.189	19.2179	180.692	3.311096E-02	57.1032	1.155107E-02	49.0900
-3.25	6.90128	316.245	18.1906	180.597	4.812628E-03	236.866	9.590541E-03	323.803
-3.35	6.32483	316.338	17.2799	180.596	2.302809E-02	37.2688	1.261047E-02	256.684
-3.45	5.80876	316.450	16.3831	180.702	1.719604E-02	133.563	1.561523E-02	184.281
-3.55	5.33714	316.514	15.5352	180.664	5.058415E-03	75.6490	1.663589E-02	143.282
-3.65	4.90105	316.817	14.7594	180.724	1.736592E-02	178.594	3.828392E-02	120.294
-3.75	4.67084	314.850	14.0427	180.908	5.914298E-02	232.746	0.195835	325.490
-3.85	4.17359	316.188	13.2905	180.739	2.301636E-03	313.302	2.195198E-02	314.440

THIS CHECKS THE ACCURACY OF THE WKB APPROXIMATION FOR AMPLITUDE

-0.15	3.138082E-02	7.92047	3.236662E-02	8.47009	0.516793	2.07989	0.339006	0.589642
-0.25	5.922882E-02	11.1052	6.133633E-02	11.6646	0.508332	2.79456	0.326011	0.741994
-0.35	9.425267E-02	15.4647	9.737104E-02	15.9544	0.502099	3.72820	0.312821	0.927719
-0.45	0.128178	21.3626	0.130981	21.6471	0.503129	4.93149	0.304038	1.15121
-0.55	0.160837	29.2307	0.161159	29.0955	0.515272	6.45618	0.302149	1.41572
-0.65	0.206855	39.5501	0.201922	38.6757	0.537146	8.34687	0.301151	1.72200
-0.75	0.284306	52.8049	0.271772	50.7417	0.559918	10.6262	0.289900	2.06631
-0.85	0.389310	69.3907	0.367296	65.5423	0.568856	13.2707	0.267622	2.43748
-0.95	0.502148	89.4587	0.467885	83.0864	0.548156	16.1750	0.255279	2.81296
-1.05	0.616328	112.676	0.564487	102.943	0.490045	19.0999	0.267485	3.15376
-1.15	0.747719	137.898	0.670910	123.968	0.413181	21.6090	0.273399	3.39867
-1.25	0.896054	162.794	0.790588	144.002	0.383971	23.0047	0.245735	3.46026
-1.35	1.01767	183.605	0.885991	159.648	0.449820	22.3195	0.225147	3.23103
-1.45	1.07354	195.529	0.916905	166.620	0.535003	18.5460	0.243327	2.62985
-1.55	1.06670	194.773	0.884078	161.749	0.554020	11.5454	0.236444	1.74931
-1.65	1.00455	182.071	0.808675	146.804	0.515203	3.45371	0.219690	0.927465
-1.75	0.896509	162.973	0.709995	128.483	0.497712	2.85220	0.221268	0.359198
-1.85	0.786800	143.522	0.621044	113.345	0.518205	5.22225	0.211331	0.508603
-1.95	0.700934	127.104	0.571644	104.344	0.526273	4.64179	0.210421	0.559931
-2.05	0.632106	115.092	0.554365	100.417	0.509886	3.07046	0.204989	0.410325
-2.15	0.587738	106.885	0.545782	99.2004	0.493143	1.71231	0.197898	0.255100
-2.25	0.556309	101.089	0.542366	98.9668	0.491560	0.836762	0.195376	0.149800
-2.35	0.530904	96.6583	0.544558	98.7839	0.498488	0.486048	0.189572	9.594051E-02
-2.45	0.512333	93.0590	0.540570	98.2347	0.500262	0.510709	0.185992	7.890003E-02
-2.55	0.494995	90.0944	0.533469	97.2237	0.495434	0.538568	0.181028	7.482007E-02
-2.65	0.482336	87.6873	0.528552	95.9085	0.493242	0.459850	0.177474	6.802170E-02
-2.75	0.471551	85.7132	0.520094	94.6785	0.495832	0.285859	0.172863	5.466322E-02
-2.85	0.461322	83.9977	0.516470	93.9541	0.496001	9.705652E-02	0.169387	4.171049E-02
-2.95	0.454019	82.4518	0.515826	93.7257	0.493996	0.170539	0.165211	4.525200E-02
-3.05	0.446022	81.1192	0.513235	93.4474	0.495236	0.236352	0.161371	5.671593E-02
-3.15	0.439300	80.0051	0.511026	92.8367	0.495757	0.159950	0.158159	5.530023E-02
-3.25	0.434453	78.9810	0.507562	92.3796	0.494705	2.444054E-02	0.154316	4.870[illegible]E-02
-3.35	0.429826	78.0214	0.507664	92.2538	0.495608	0.122942	0.150444	6.732471E-02
-3.45	0.425272	77.2361	0.505419	91.9505	0.494938	9.651315E-02	0.146964	8.764082E-02
-3.55	0.420579	76.4923	0.504207	91.6623	0.495020	2.984607E-02	0.143916	9.815641E-02
-3.65	0.415387	75.7130	0.503506	91.5495	0.494429	0.107717	0.141110	0.237467
-3.75	0.440584	77.7764	0.503878	91.5697	0.491345	0.385661	0.142475	1.27701
-3.85	0.412688	74.9093	0.500803	91.1082	0.494586	1.577806E-02	0.134395	0.150484

***ELIMINATES THE DOWN-GOING MAGNETIC WAVE ***

-0.15	2.842171E-14	180.000	1.136310E-03	89.4602	0.486117	254.057	0.313394	209.743
-0.25	2.273737E-13	270.000	1.106545E-03	40.3430	0.461649	253.776	0.289329	209.610
-0.35	2.343714E-13	255.964	1.057743E-03	347.620	0.437405	253.428	0.267238	209.498
-0.45	6.821210E-13	90.0000	1.007895E-03	291.102	0.413302	253.103	0.246955	209.407
-0.55	2.273737E-13	270.000	9.715651E-04	230.811	0.389628	252.899	0.228323	209.338
-0.65	0.	90.0000	9.602382E-04	167.115	0.366982	252.893	0.211191	209.290
-0.75	2.842171E-14	0.	9.805569E-04	100.784	0.346108	253.110	0.195420	209.262
-0.85	1.819211E-12	269.105	1.031165E-03	32.7261	0.327614	253.486	0.180884	209.250
-0.95	1.850688E-12	100.620	1.101390E-03	323.516	0.311649	253.870	0.167475	209.248
-1.05	1.824532E-12	94.4672	1.173660E-03	253.110	0.297705	254.076	0.155096	209.253
-1.15	1.845661E-12	99.7524	1.228322E-03	180.898	0.284724	253.984	0.143668	209.257
-1.25	4.547474E-13	0.	1.249388E-03	105.828	0.271550	253.622	0.133123	209.257
-1.35	1.833145E-12	262.875	1.231469E-03	26.4427	0.257555	253.187	0.123398	209.253
-1.45	4.547474E-13	180.000	1.188803E-03	300.925	0.243090	252.953	0.114432	209.248
-1.55	1.818989E-12	0.	1.163719E-03	207.763	0.229363	253.088	0.106161	209.246
-1.65	2.145039E-12	237.995	1.216281E-03	107.666	0.217546	253.465	9.851823E-02	209.251
-1.75	9.094947E-13	180.000	1.363799E-03	4.02128	0.207665	253.726	9.144303E-02	209.262
-1.85	4.547474E-13	0.	1.517956E-03	258.522	0.198528	253.623	8.488834E-02	209.275
-1.95	0.	90.0000	1.540212E-03	148.486	0.188902	253.278	7.881686E-02	209.283
-2.05	8.198074E-13	123.690	1.383049E-03	29.8019	0.178677	253.080	7.319607E-02	209.286
-2.15	2.273737E-13	270.000	1.156249E-03	260.715	0.169071	253.270	6.799253E-02	209.290
-2.25	0.	90.0000	9.358296E-04	126.591	0.161076	253.545	6.316794E-02	209.300
-2.35	6.431099E-13	135.000	6.211934E-04	354.732	0.153876	253.457	5.868784E-02	209.311
-2.45	6.431099E-13	135.000	2.354644E-04	236.244	0.146140	253.185	5.452951E-02	209.317
-2.55	2.273737E-13	90.0000	2.019997E-04	181.313	0.138342	253.235	5.067465E-02	209.322
-2.65	2.273737E-13	0.	4.075890E-04	35.4142	0.131703	253.452	4.709783E-02	209.331
-2.75	0.	90.0000	4.933262E-04	232.271	0.125630	253.342	4.377273E-02	209.342
-2.85	0.	90.0000	3.342895E-04	56.0284	0.119169	253.207	4.068431E-02	209.348
-2.95	2.273737E-13	0.	1.003255E-04	217.105	0.113198	253.361	3.781867E-02	209.356
-3.05	2.273737E-13	180.000	1.577546E-04	203.724	0.107916	253.337	3.515410E-02	209.370
-3.15	2.273737E-13	90.0000	2.649305E-04	353.357	0.102470	253.244	3.267695E-02	209.380
-3.25	2.273737E-13	90.0000	2.048078E-04	110.888	9.743569E-02	253.345	3.037780E-02	209.394
-3.35	0.	90.0000	1.079772E-04	156.768	9.275136E-02	253.281	2.823810E-02	209.418
-3.45	0.	90.0000	2.105766E-04	208.294	8.811819E-02	253.299	2.625088E-02	209.445
-3.55	2.428352E-13	110.556	2.855947E-04	263.367	8.387057E-02	253.292	2.440560E-02	209.511
-3.65	2.842171E-13	126.870	4.240028E-04	284.014	7.971184E-02	253.309	2.269789E-02	209.689
-3.75	1.136868E-13	0.	2.398317E-03	147.572	7.562167E-02	253.297	2.158209E-02	207.750
-3.85	1.421085E-13	180.000	2.668200E-04	113.871	7.215709E-02	253.281	1.960397E-02	209.083